

FWF Österreichischer Wissenschaftsfonds



NAWI Graz
Natural Sciences

fct
Fundação
para a Ciência
e a Tecnologia

ExoHad
Exotic Hadrons Topical Collaboration

The few-body problem in QCD

(with functional methods)

Gernot Eichmann
University of Graz

2025 International Workshop and School on Hadron Structure and Strong Interactions

Nanjing, China, October 13-17, 2025

1. **Introduction**
2. **Spectrum & symmetries**
3. **Quark models**
4. **QFT toolbox**
5. **Functional methods**
6. **n-point functions & tensor bases**
7. **Multiquark states**
8. **Light-front wave functions**

Lecture notes "QCD and hadron physics",
https://particle.uni-graz.at/en/quarks-hadrons-and-nuclei

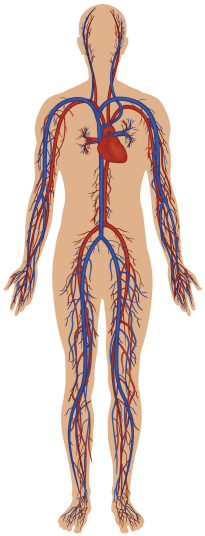
GE, "*Hadron physics with functional methods*",
arXiv: 2503.10397 (for Encyclopedia of Particle Physics)

Burkert, GE, Klempt, "*The impact of* γN *and* γ*N *interactions on our understanding of nucleon excitations*", arXiv:2506.16482

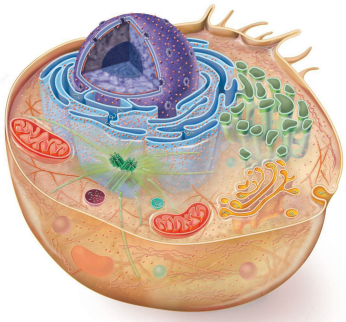
GE, Sanchis-Alepuz, Williams, Alkofer, Fischer,
"*Baryons as relativistic three-quark bound states*",
Prog. Part. Nucl. Phys. 91 (2016), arXiv: 1606.09602

Barabanov, Bedolla, Brooks, Cates, Chen et al.,
"*Diquark correlations in hadron physics: Origin, impact and evidence*", Prog. Part. Nucl. Phys. 116 (2021),
arXiv: 2008.07630

QCD binds us together



1 m



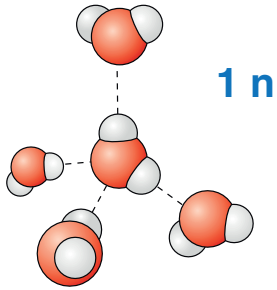
10 µm

Biology



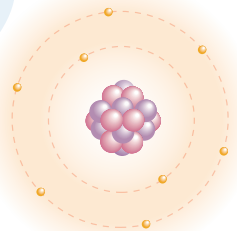
10 nm

Chemistry



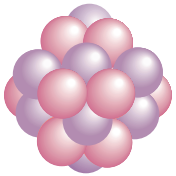
1 nm

Atomic physics



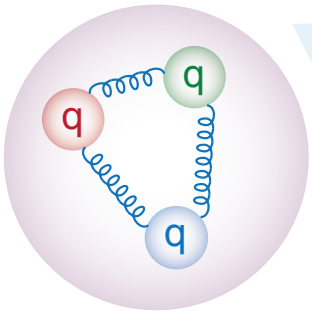
1 Å

Nuclear physics



10 fm

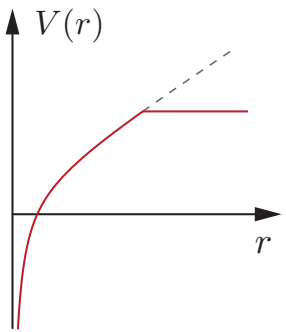
Quantum Chromo-Dynamics



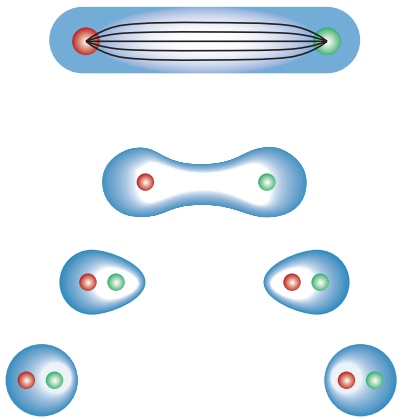
1 fm

Confinement

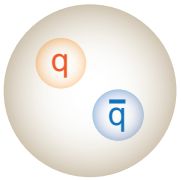
- Quarks & gluons are **confined** in hadrons, cannot be observed in isolation



Microscopic origin still unclear – $1M prize

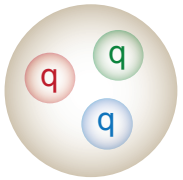


- Only colorless **bound states** observable:



$3 \times \bar{3} = \mathbf{1} + 8$

Mesons



$3 \times 3 \times 3 = \mathbf{1} + 8 + 8 + 10$

Baryons

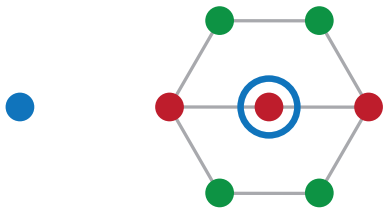
The hadron zoo

http://pdglive.lbl.gov

Mesons

$J^{P(C)}$

0^{-+}	0^{++}	1^{-+}	1^{--}	1^{++}	1^{+-}	2^{-+}	2^{++}	3^{--}
$\pi(140)$	$a_0(980)$	$\pi_1(1400)$	$\rho(770)$	$a_1(1260)$	$b_1(1235)$	$\pi_2(1670)$	$a_2(1320)$	$\rho_3(1690)$
$\pi(1300)$	$a_0(1450)$	$\pi_1(1600)$	$\rho(1450)$	$a_1(1420)$		$\pi_2(1880)$	$a_2(1700)$	$\rho_3(1990)$
$\pi(1800)$	$a_0(1950)$		$\rho(1570)$	$a_1(1640)$				
			$\rho(1700)$					
			$\rho(1900)$					
$K(494)$	$K_0^*(800)$		$K^*(892)$	$K_1(1400)$	$K_1(1270)$	$K_2(1580)$	$K_2^*(1430)$	$K_3^*(1780)$
$K(1460)$	$K_0^*(1430)$		$K^*(1410)$	$K_1(1650)$		$K_2(1770)$	$K_2^*(1980)$	
$K(1830)$	$K_0^*(1950)$		$K^*(1680)$			$K_2(1820)$		
$\eta(548)$	$f_0(500)$		$\omega(782)$	$f_1(1285)$	$h_1(1170)$	$\eta_2(1645)$	$f_2(1270)$	$\omega_3(1670)$
$\eta'(958)$	$f_0(980)$		$\phi(1020)$	$f_1(1420)$	$h_1(1380)$	$\eta_2(1870)$	$f_2(1430)$	$\phi_3(1850)$
$\eta(1295)$	$f_0(1370)$		$\omega(1420)$	$f_1(1510)$	$h_1(1595)$		$f_2'(1525)$	
$\eta(1405)$	$f_0(1500)$		$\omega(1650)$				$f_2(1565)$	
$\eta(1475)$	$f_0(1710)$		$\phi(1680)$				$f_2(1640)$	
$\eta(1760)$							$f_2(1810)$	
							$f_2(1910)$	
							$f_2(1950)$	

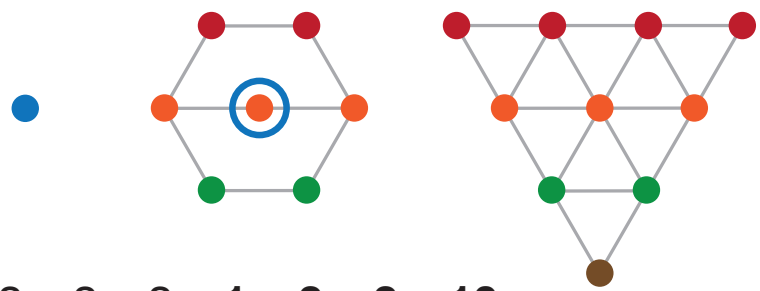


$3 \times \bar{3} = \mathbf{1} + \mathbf{8}$

Baryons

J^P

$\frac{1}{2}^+$	$\frac{1}{2}^-$	$\frac{3}{2}^+$	$\frac{3}{2}^-$	$\frac{5}{2}^+$	$\frac{5}{2}^-$	$\frac{7}{2}^+$
$N(939)$	$N(1535)$	$N(1720)$	$N(1520)$	$N(1680)$	$N(1675)$	$N(1990)$
$N(1440)$	$N(1650)$	$N(1900)$	$N(1700)$	$N(1860)$		
$N(1710)$	$N(1895)$		$N(1875)$	$N(2000)$		
$N(1880)$						
$\Delta(1910)$	$\Delta(1620)$	$\Delta(1232)$	$\Delta(1700)$	$\Delta(1905)$	$\Delta(1930)$	$\Delta(1950)$
	$\Delta(1900)$	$\Delta(1600)$	$\Delta(1940)$	$\Delta(2000)$		
		$\Delta(1920)$				
$\Lambda(1116)$	$\Lambda(1405)$	$\Lambda(1890)$	$\Lambda(1520)$	$\Lambda(1820)$	$\Lambda(1830)$	
$\Lambda(1600)$	$\Lambda(1670)$		$\Lambda(1690)$			
$\Lambda(1810)$	$\Lambda(1800)$					
$\Sigma(1189)$	$\Sigma(1750)$	$\Sigma(1385)$	$\Sigma(1670)$	$\Sigma(1915)$	$\Sigma(1775)$	
$\Sigma(1660)$			$\Sigma(1940)$			
$\Sigma(1880)$						
$\Xi(1315)$		$\Xi(1530)$	$\Xi(1820)$			
		$\Omega(1672)$				



$3 \times 3 \times 3 = \mathbf{1} + \mathbf{8} + \mathbf{8} + \mathbf{10}$

The hadron zoo

http://pdglive.lbl.gov

Mesons

$J^{P(C)}$

0^{-+}

$\pi(140)$ $\pi(1300)$ $\pi(1800)$ $a_0(980)$ $a_0(1450)$ $a_0(1950)$ $\pi_1(1400)$ $\pi_1(1600)$

1^{--}

$\rho(770)$ $\rho(1450)$ $\rho(1570)$ $\rho(1700)$ $\rho(1900)$ $a_1(1260)$ $a_1(1420)$ $a_1(1640)$ $b_1(1235)$ $\pi_2(1670)$ $\pi_2(1880)$ $a_2(1320)$ $a_2(1700)$ $\rho_3(1690)$ $\rho_3(1990)$

$K(494)$ $K(1460)$ $K(1830)$ $K_0^*(800)$ $K_0^*(1430)$ $K_0^*(1950)$ $K^*(892)$ $K^*(1410)$ $K^*(1680)$ $K_1(1400)$ $K_1(1650)$ $K_1(1270)$ $K_2(1580)$ $K_2(1770)$ $K_2(1820)$ $K_2^*(1430)$ $K_3^*(1780)$ $K_3^*(1980)$

$\eta(548)$ $\eta'(958)$ $\eta(1295)$ $\eta(1405)$ $\eta(1475)$ $\eta(1760)$ $f_0(500)$ $f_0(980)$ $f_0(1370)$ $f_0(1500)$ $f_0(1710)$ $\omega(782)$ $\phi(1020)$ $\omega(1420)$ $\omega(1650)$ $\phi(1680)$ $f_1(1285)$ $f_1(1420)$ $f_1(1510)$ $h_1(1170)$ $h_1(1380)$ $h_1(1595)$ $\eta_2(1645)$ $\eta_2(1870)$ $f_2(1270)$ $f_2(1430)$ $f_2'(1525)$ $f_2(1565)$ $f_2(1640)$ $f_2(1810)$ $f_2(1910)$ $f_2(1950)$ $\omega_3(1670)$ $\phi_3(1850)$

Baryons

J^P

$\frac{1}{2}^+$ $N(939)$ $N(1440)$ $N(1710)$ $N(1880)$

$\frac{1}{2}^-$ $N(1535)$ $N(1650)$ $N(1895)$

$\frac{3}{2}^+$ $N(1720)$ $N(1900)$

$\frac{3}{2}^-$ $N(1520)$ $N(1700)$ $N(1875)$

$\frac{5}{2}^+$ $N(1680)$ $N(1860)$ $N(2000)$

$\frac{5}{2}^-$ $N(1675)$

$\frac{7}{2}^+$ $N(1990)$

~1 GeV

$\Delta(1910)$ $\Delta(1620)$ $\Delta(1900)$ $\Delta(1232)$ $\Delta(1600)$ $\Delta(1920)$ $\Delta(1700)$ $\Delta(1940)$ $\Delta(1905)$ $\Delta(2000)$ $\Delta(1930)$ $\Delta(1950)$

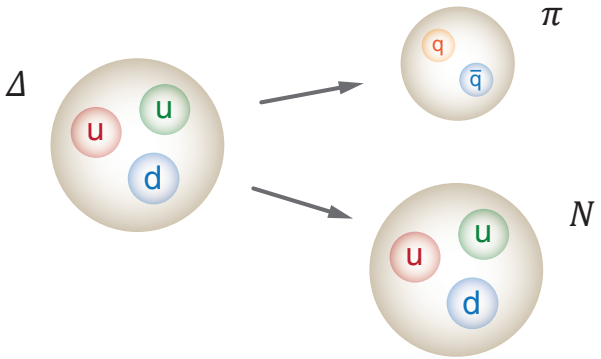
$\Lambda(1116)$ $\Lambda(1600)$ $\Lambda(1810)$ $\Lambda(1405)$ $\Lambda(1670)$ $\Lambda(1800)$ $\Lambda(1890)$ $\Lambda(1520)$ $\Lambda(1690)$ $\Lambda(1820)$ $\Lambda(1830)$

$\Sigma(1189)$ $\Sigma(1660)$ $\Sigma(1880)$ $\Sigma(1750)$ $\Sigma(1385)$ $\Sigma(1670)$ $\Sigma(1940)$ $\Sigma(1915)$ $\Sigma(1775)$

$\Xi(1315)$ $\Xi(1530)$ $\Xi(1820)$

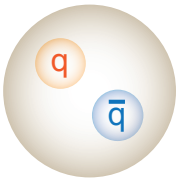
$\Omega(1672)$

Most hadrons are **resonances** and decay into other hadrons

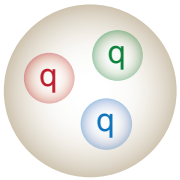


And it doesn't stop there ...

Familiar states

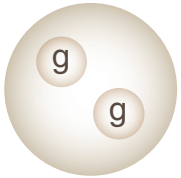


Mesons

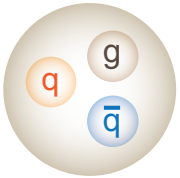


Baryons

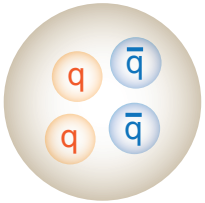
Exotic states



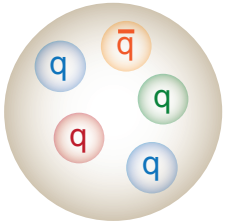
Glueballs?



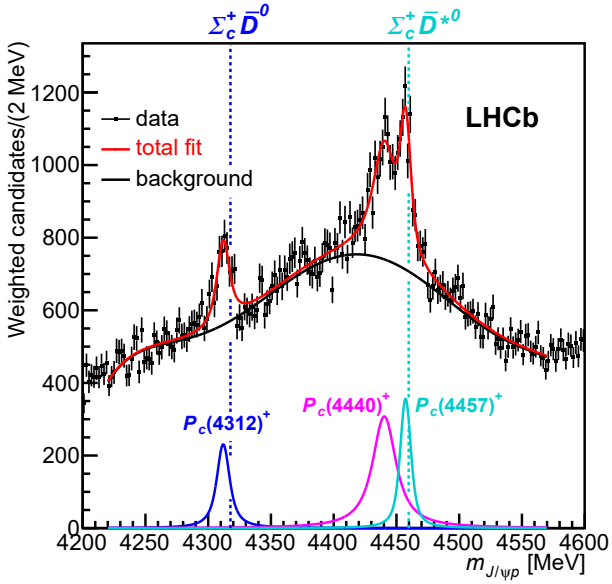
Hybrid mesons?



Tetraquarks?



Pentaquarks?



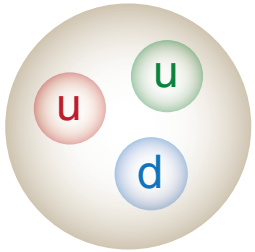
Aaij et al., Phys. Rev. Lett. 112 (2019) 222001

To understand QCD, we must understand
spectrum and interactions of hadrons!

Mass generation

Three current quarks
do **not** make a proton:

$2m_u + m_d$ ~ 10 MeV << 1 GeV ?



Higgs: Current-quark masses

QCD: **Mass generation** from
quark-gluon interactions,
“constituent-quark masses”

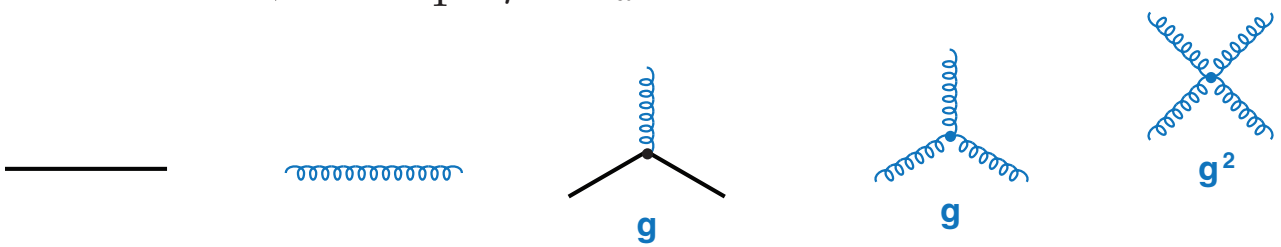
u	d	s	c	b	t
3	5	100	1000	4000	175000
350	350	350	350	350	350

[MeV]



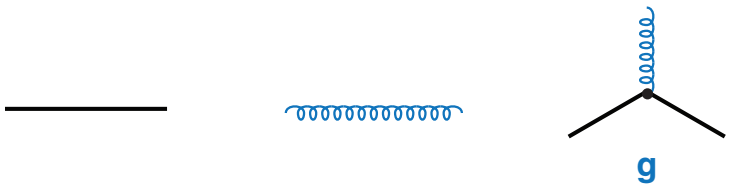
QCD Lagrangian

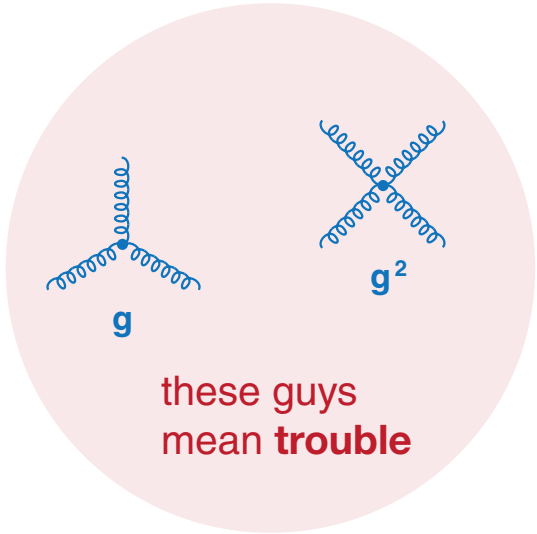
$$\mathcal{L} = \bar{\psi}\left(\not{\partial} + i\,g\not{A} + m\right)\psi + \frac{1}{4}\,F^{a}_{\mu\nu}\,F^{\mu\nu}_{a}$$



QCD Lagrangian

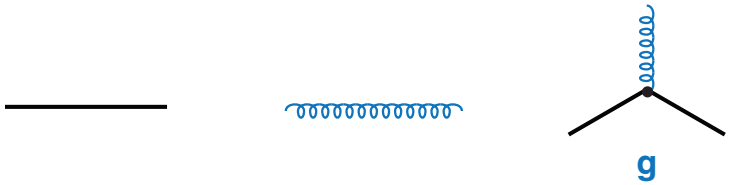
$$\mathcal{L} = \bar{\psi}\left(\not{\partial} + i\,g\not{A} + m\right)\psi + \frac{1}{4}\,F^{a}_{\mu\nu}\,F^{\mu\nu}_{a}$$

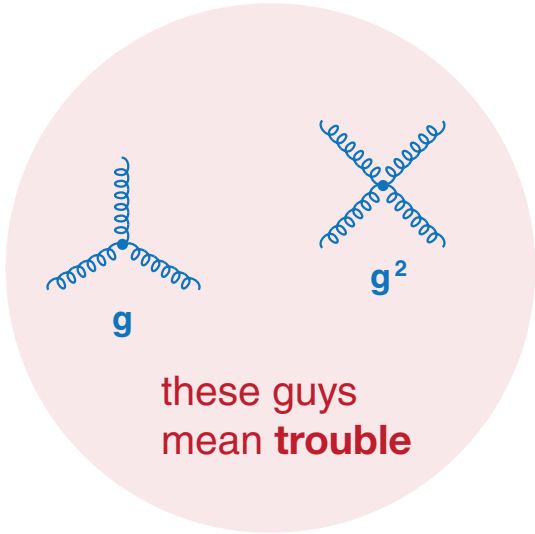




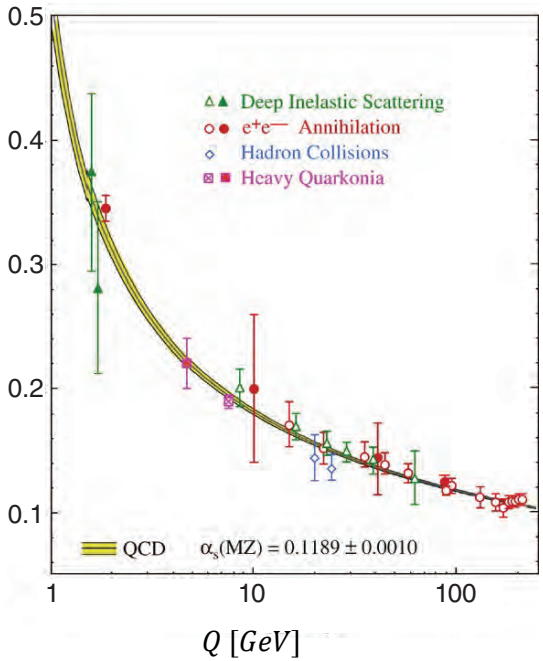
QCD Lagrangian

$$\mathcal{L} = \bar{\psi}\left(\not{\partial} + i\,g\not{A} + m\right)\psi + \frac{1}{4}\,F^{a}_{\mu\nu}\,F^{\mu\nu}_{a}$$





$$\alpha = \frac{g^2}{4\pi}$$



At small momenta, coupling becomes large → we need **nonperturbative methods!**

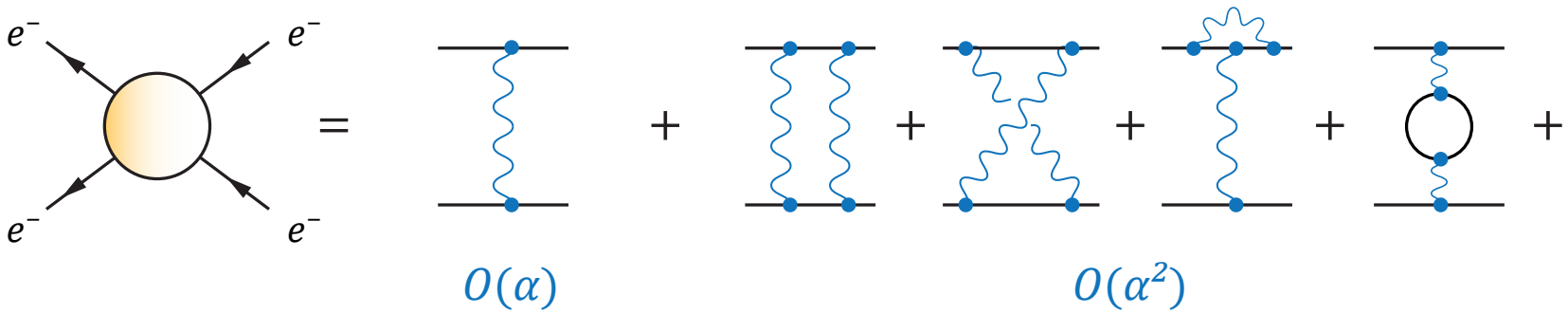
At large momenta, quarks & gluons behave as quasi-free particles: **asymptotic freedom**



Gross, Politzer, Wilczek 1973

Perturbative vs. nonperturbative

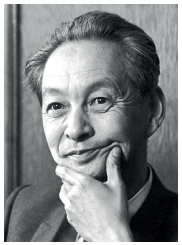
Interaction between **two electrons**:

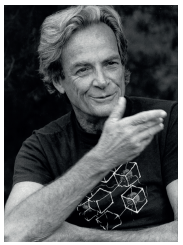


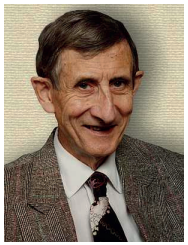
"Quantum mechanics"

Perturbative QFT:
Feynman diagrams & loops


J. S. Schwinger

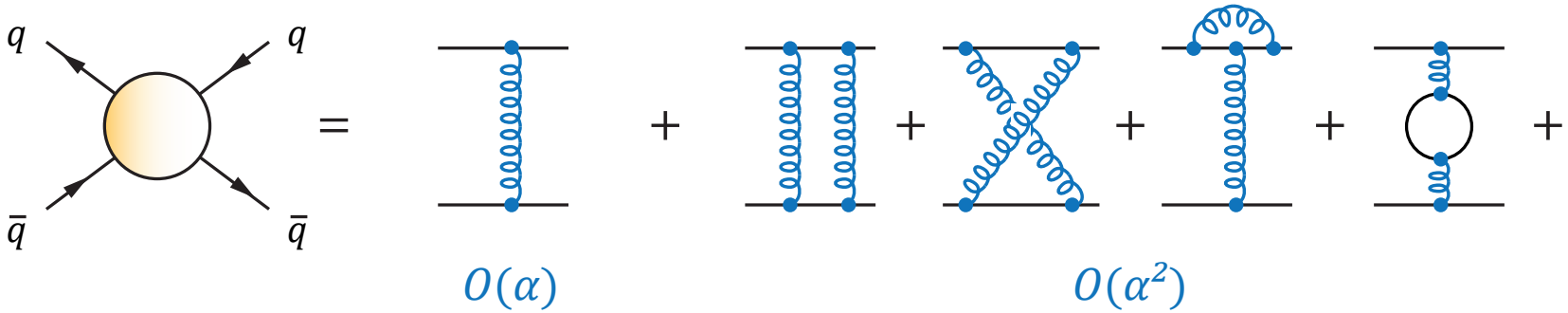

S. Tomonaga


R. P. Feynman

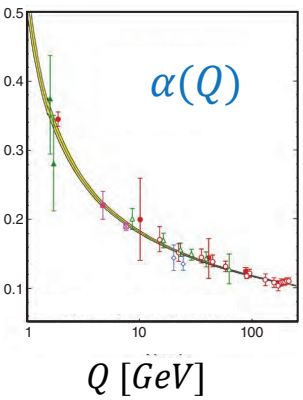

F. J. Dyson

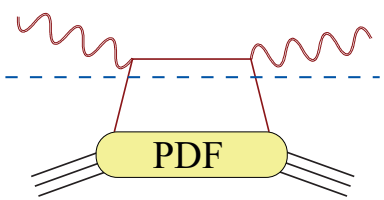
Perturbative vs. nonperturbative

Interaction between **quark and antiquark?**



Answer still correct at large momenta:
high-energy scattering experiments



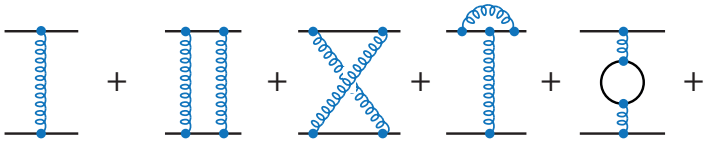


What about low momenta, confinement, mass generation, hadron physics?

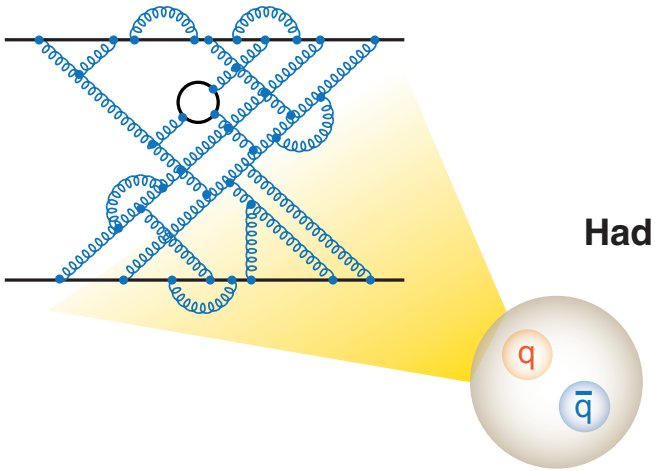
→ need **nonperturbative methods!**

Emergent phenomena

Hadron physics:

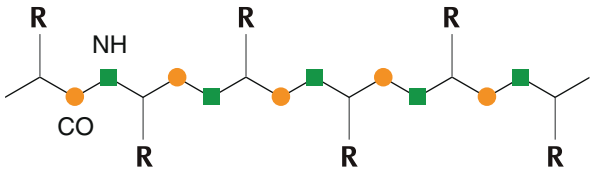


Perturbation theory

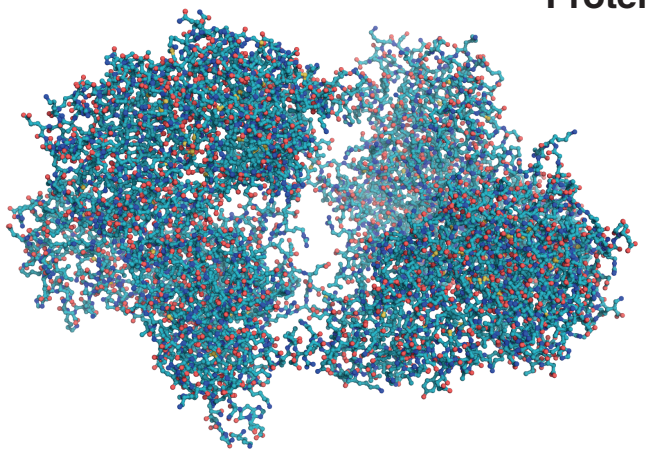


Hadron

Biology:



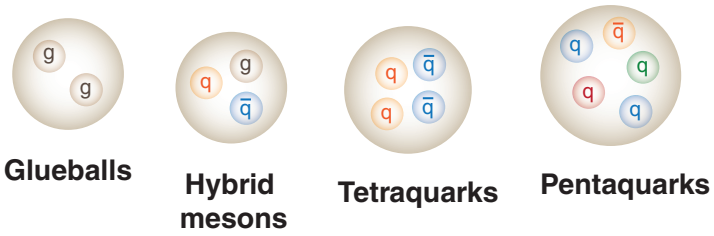
Protein



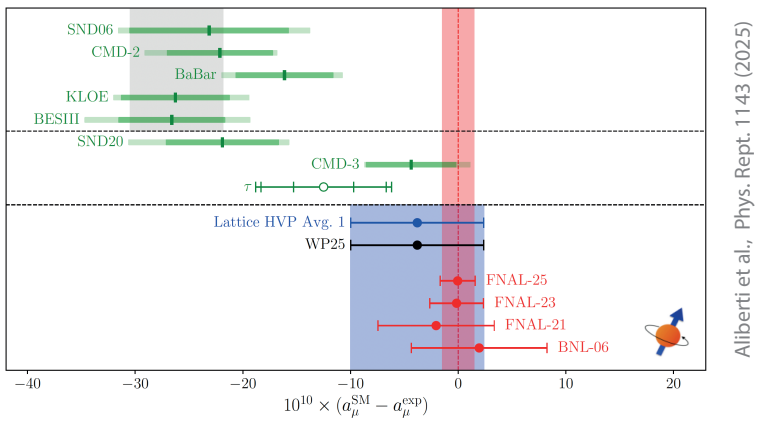
Tim Vickers, Public Domain

Many open questions!

- **Understanding exotic hadrons:**
  Hadron spectroscopy at LHC, Belle II, BES III, PANDA, JLab, ELSA, ...



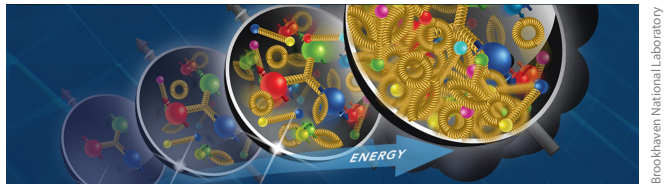
- **QCD contributions to BSM searches,**
  e.g. muon g-2 at Fermilab



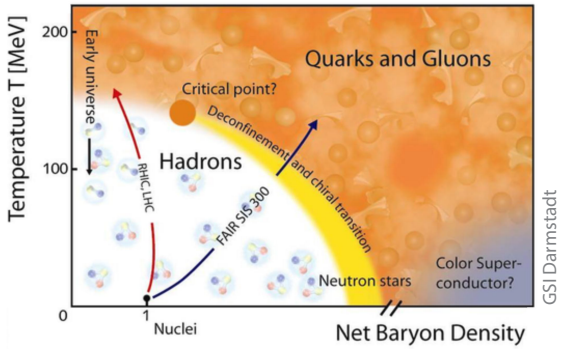
- **Mass generation and confinement?**



- **Quark-gluon structure of hadrons and nuclei:**
  EIC, JLab, COMPASS/AMBER, ...

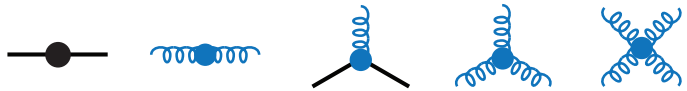


- **The phases of QCD:** LHC, RHIC, CBM, ...

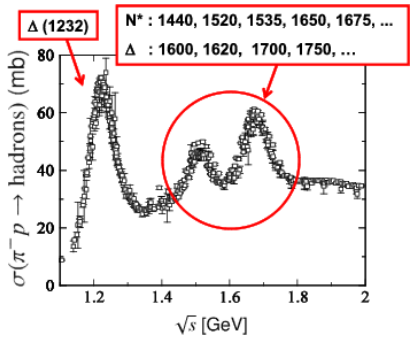


Theory tools

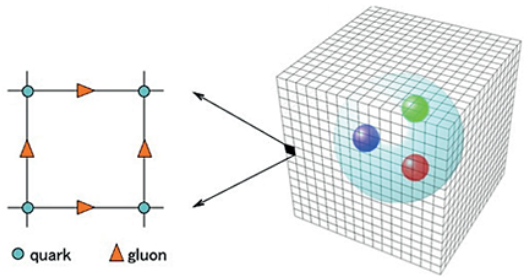
Functional methods
(Dyson-Schwinger & Bethe-Salpeter eqs, FRG, ...)



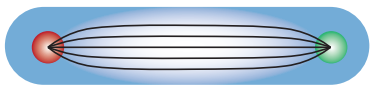
Amplitude analyses



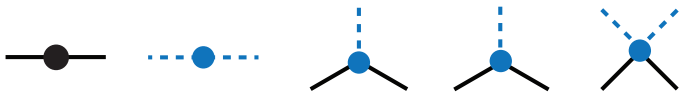
Lattice QCD



Phenomenological models
(Quark models, ...)



Effective theories
(Chiral perturbation theory, ...)



1. Introduction

2. Spectrum & symmetries

3. Quark models

4. QFT toolbox

5. Functional methods

6. n-point functions & tensor bases

7. Multiquark states

8. Light-front wave functions

Lecture notes "QCD and hadron physics",
https://particle.uni-graz.at/en/quarks-hadrons-and-nuclei

GE, "*Hadron physics with functional methods*",
arXiv: 2503.10397 (for Encyclopedia of Particle Physics)

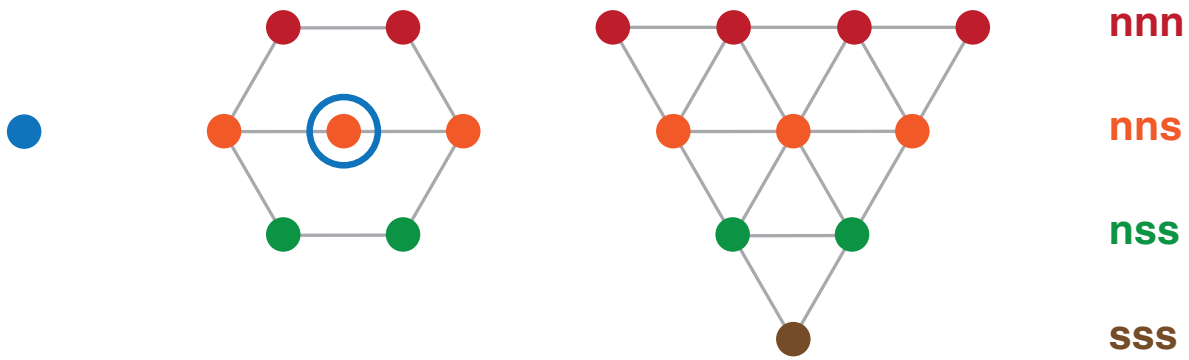
Burkert, GE, Klempt, "*The impact of* γN *and* γ*N *interactions on our understanding of nucleon excitations*", arXiv:2506.16482

GE, Sanchis-Alepuz, Williams, Alkofer, Fischer,
"*Baryons as relativistic three-quark bound states*",
Prog. Part. Nucl. Phys. 91 (2016), arXiv: 1606.09602

Barabanov, Bedolla, Brooks, Cates, Chen et al.,
"*Diquark correlations in hadron physics: Origin, impact and evidence*", Prog. Part. Nucl. Phys. 116 (2021),
arXiv: 2008.07630

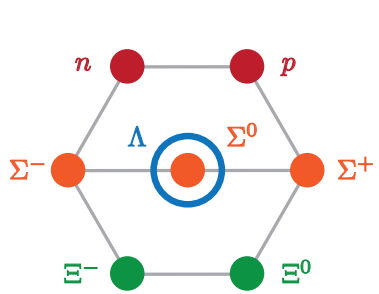
Symmetries in the spectrum

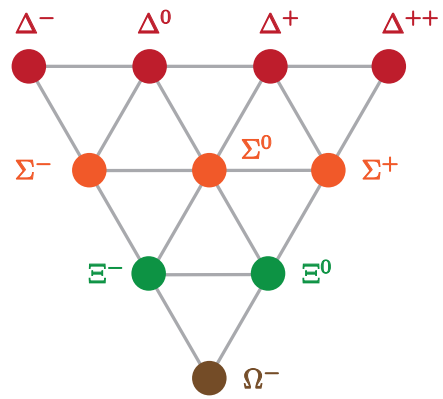
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$N(1440)$	$N(1650)$	$N(1900)$	$N(1700)$	$N(1860)$		
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$N(1880)$						
$\Delta(1910)$	$\Delta(1620)$	$\Delta(1232)$	$\Delta(1700)$	$\Delta(1905)$	$\Delta(1930)$	$\Delta(1950)$
	$\Delta(1900)$	$\Delta(1600)$	$\Delta(1940)$	$\Delta(2000)$		
		$\Delta(1920)$				
$\Lambda(1116)$	$\Lambda(1405)$	$\Lambda(1890)$	$\Lambda(1520)$	$\Lambda(1820)$	$\Lambda(1830)$	
$\Lambda(1600)$	$\Lambda(1670)$		$\Lambda(1690)$			
$\Lambda(1810)$	$\Lambda(1800)$					
$\Sigma(1189)$	$\Sigma(1750)$	$\Sigma(1385)$	$\Sigma(1670)$	$\Sigma(1915)$	$\Sigma(1775)$	
$\Sigma(1660)$			$\Sigma(1940)$			
$\Sigma(1880)$						
$\Xi(1315)$		$\Xi(1530)$	$\Xi(1820)$			
		$\Omega(1672)$				



Symmetries in the spectrum

$\frac{1}{2}^+$	$\frac{1}{2}^-$	$\frac{3}{2}^+$	$\frac{3}{2}^-$	$\frac{5}{2}^+$	$\frac{5}{2}^-$	$\frac{7}{2}^+$
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nnn

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Symmetries of QCD

$$S = \int d^4x \, \mathcal{L} \qquad \mathcal{L} = \bar{\psi} \left(i \partial\!\!\!/ + g A\!\!\!/ - M \right) \psi - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a$$

Quark mass matrix

$\psi_{\alpha,i,f}(x)$

Flavor: 1 ... N_f
Color: 1 ... 3
Dirac: 1 ... 4

Symmetries of QCD

$$S = \int d^4x \, \mathcal{L} \qquad \mathcal{L} = \bar{\psi} \left(i \partial\!\!\!/ + g A\!\!\!/ - M \right) \psi - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a$$

Quark mass matrix

$\psi_{\alpha,i,f}(x)$

Flavor: 1 ... N_f
Color: 1 ... 3
Dirac: 1 ... 4

- invariant under **local SU(3) gauge transformations**

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10}_S \oplus \mathbf{8}_{M_A} \oplus \mathbf{8}_{M_S} \oplus \mathbf{1}_A$$

Decuplet **Octet** **Singlet**

⇒ Baryons are **color singlets**

Symmetries of QCD

$$S = \int d^4x \, \mathcal{L} \qquad \mathcal{L} = \bar{\psi} \left(i \partial\!\!\!/ + g A\!\!\!/ - M \right) \psi - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a$$

Quark mass matrix

$\psi_{\alpha,i,f}(x)$

Flavor: 1 ... N_f
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- invariant under local SU(3) gauge transformations
- invariant under **Poincaré group**

Translations, rotations, boosts;

Two Casimir operators which label states:
M (mass), J (total angular momentum, "spin")

Symmetries of QCD

$$S = \int d^4x \, \mathcal{L} \qquad \mathcal{L} = \bar{\psi} \left(i \partial\!\!\!/ + g A\!\!\!/ - M \right) \psi - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a$$

Quark mass matrix

$\psi_{\alpha,i,f}(x)$

Flavor: 1 ... N_f
Color: 1 ... 3
Dirac: 1 ... 4

- invariant under local SU(3) gauge transformations
- invariant under Poincaré group
- invariant under C, **P**, T

Parity labels states → M, J^P

Symmetries of QCD

$$S = \int d^4x\, \mathcal{L} \qquad \mathcal{L} = \bar{\psi}\,(\, i\partial\!\!\!/ + g A\!\!\!/ - M\,)\,\psi - \tfrac{1}{4}\, F^a_{\mu\nu}\, F^{\mu\nu}_a$$

Quark mass matrix

$\psi_{\alpha,i,f}(x)$

Flavor: 1 ... N_f
Color: 1 ... 3
Dirac: 1 ... 4

- invariant under local SU(3) gauge transformations
- invariant under Poincaré group
- invariant under C, P, T
- various global flavor (non-)symmetries:

$U(1)_V$ conserved: **baryon number** $B = \frac{1}{3}\,(n_u + n_d + n_s)$

n_q ... (# quarks) – (# antiquarks)

$\psi' = e^{\,i\varepsilon}\psi$

$\Rightarrow$ Baryons carry $B = 1$ (pentaquarks too)

Symmetries of QCD

$$S = \int d^4x\, \mathcal{L} \qquad \mathcal{L} = \bar{\psi}\,(\, i\partial\!\!\!/ + g A\!\!\!/ - M\,)\,\psi - \tfrac{1}{4}\, F^a_{\mu\nu}\, F^{\mu\nu}_a$$

Quark mass matrix

$\psi_{\alpha,i,f}(x)$

- invariant under local SU(3) gauge transformations
- invariant under Poincaré group
- invariant under C, P, T
- various global flavor (non-)symmetries:

$U(1)_V$	conserved (baryon number)
$SU(N_f)_V$	if M = m (all quark masses equal)

$\psi' = e^{\,i\sum_{a=1}^{8} \varepsilon_a \mathtt{t}_a}\psi$ for 3 flavors

Define **vector currents** and charges:

$$V^\mu_a = \bar{\psi}\,\gamma^\mu\,\mathtt{t}_a\,\psi\,, \qquad \partial_\mu\, V^\mu_a = i\bar{\psi}\,[\mathsf{M},\mathtt{t}_a]\,\psi\,,$$

$$Q^V_a(t) = \int d^3x\, \psi^\dagger\,\mathtt{t}_a\,\psi$$

Noether theorem: if symmetry preserved, divergence of current vanishes and charge is conserved. Can only work if M = m.

But the diagonal SU(3) generators $\mathtt{t}_3\,, \mathtt{t}_8$ commute with M → always conserved!
Isospin I_3, **hypercharge** Y**:**

$$I_3 = \tfrac{1}{2}\,(n_u - n_d), \qquad Y = \tfrac{1}{3}\,(n_u + n_d - 2n_s)$$

These are still good quantum numbers, even if flavor symmetry is broken!

Symmetries of QCD

$$S = \int d^4x\, \mathcal{L} \qquad \mathcal{L} = \bar{\psi}\,(\, i\partial\!\!\!/ + g A\!\!\!/ - M\,)\,\psi - \tfrac{1}{4}\, F^a_{\mu\nu}\, F^{\mu\nu}_a$$

Quark mass matrix

$\psi_{\alpha,i,f}(x)$

- invariant under local SU(3) gauge transformations
- invariant under Poincaré group
- invariant under C, P, T
- various global flavor (non-)symmetries:

chiral symmetry

$U(1)_V$	conserved (baryon number)
$SU(N_f)_V$	if M = m (all quark masses equal)
$SU(N_f)_A$	if M = 0 (but: spontaneously broken)

$\psi' = e^{\,i\gamma_5 \sum_{a=1}^{8} \varepsilon_a \mathsf{t}_a}\psi$ for 3 flavors

Define **axialvector currents** and charges:

$$A^\mu_a = \bar{\psi}\,\gamma^\mu\gamma_5\,\mathsf{t}_a\,\psi\,, \qquad \partial_\mu A^\mu_a = i\bar{\psi}\,\{\mathsf{M},\mathsf{t}_a\}\,\gamma_5\,\psi\,,$$

$$Q^A_a(t) = \int d^3x\, \psi^\dagger\gamma_5\,\mathsf{t}_a\,\psi$$

Symmetry only if M = 0 (**chiral limit**)

But symmetry spontaneously broken in QFT: **dynamical chiral symmetry breaking**

Symmetries of QCD

$$S = \int d^4x\, \mathcal{L} \qquad \mathcal{L} = \bar{\psi}\,(\, i\partial\!\!\!/ + g A\!\!\!/ - M\,)\,\psi - \tfrac{1}{4}\, F^a_{\mu\nu}\, F^{\mu\nu}_a$$

Quark mass matrix

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$U(1)_V$	conserved (baryon number)
$SU(N_f)_V$	if M = m (all quark masses equal)
$SU(N_f)_A$	if M = 0 (but: spontaneously broken)
$U(1)_A$	if M = 0 (but: anomalously broken)

Only M = 0, but **anomalous breaking** in QFT (symmetry does not survive quantization)

$$\psi' = e^{\,i\gamma_5\,\varepsilon}\,\psi$$

Symmetries of QCD

$$S = \int d^4x\, \mathcal{L} \qquad \mathcal{L} = \bar{\psi}\,(\, i\partial\!\!\!/ + g A\!\!\!/ - M\,)\,\psi - \tfrac{1}{4}\, F^a_{\mu\nu}\, F^{\mu\nu}_a$$

Quark mass matrix

$\psi_{\alpha,i,f}(x)$

- invariant under local SU(3) gauge transformations
- invariant under Poincaré group
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- various global flavor (non-)symmetries:

	$U(1)_V$	conserved (baryon number)
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	$U(1)_A$	if M = 0 (but: anomalously broken)

$$\psi' = e^{\,i\gamma_5\,\varepsilon}\,\psi$$

⇒ **Quantum numbers** for labelling baryons:

$$M,\ J^P,\ I_3,\ Y$$

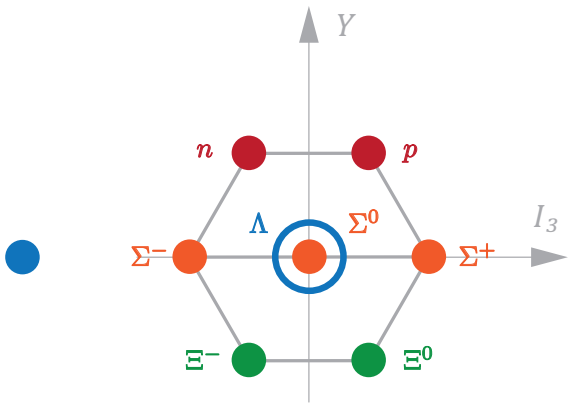
Symmetries of QCD

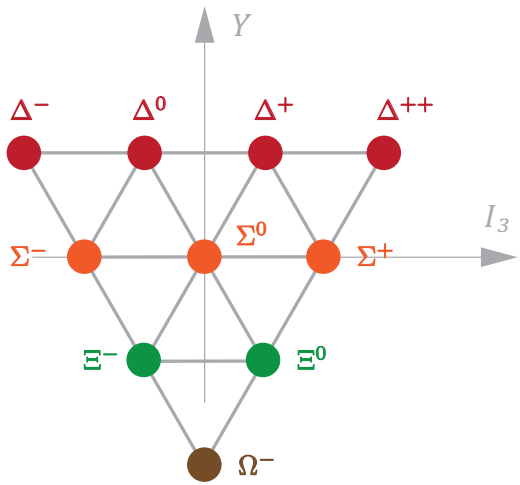
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$\Xi(1315)$		$\Xi(1530)$ $\Omega(1672)$	$\Xi(1820)$			

Quantum numbers
for labelling baryons:

$$M,\ J^P,\ I_3,\ Y$$

Symmetries of the spectrum are the symmetries of $\mathcal{L}$!





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Symmetries of QCD

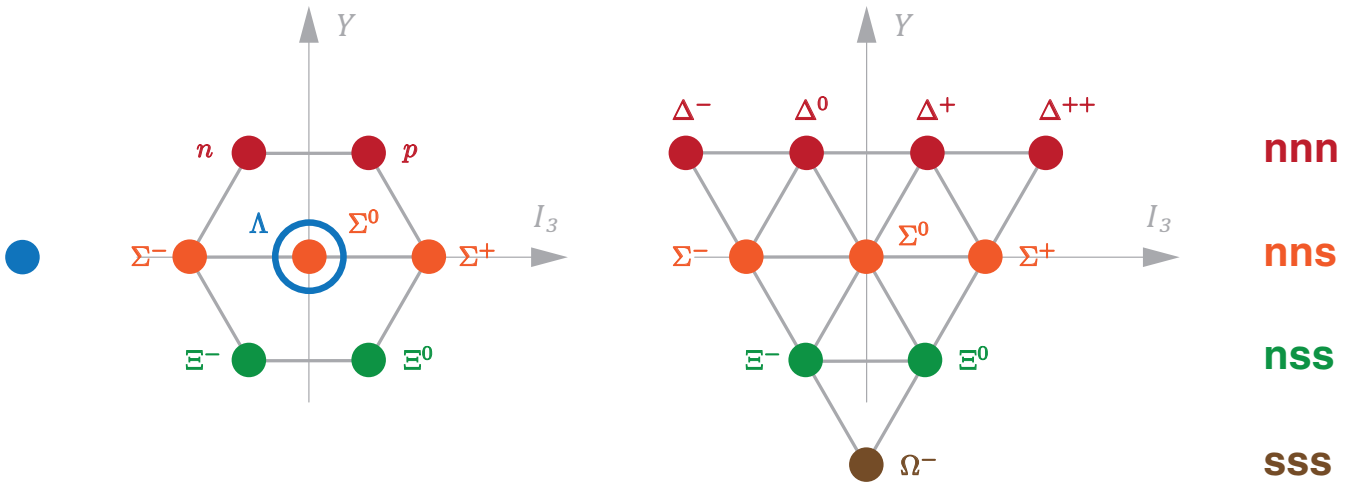
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$SU(3)_f$ is broken:

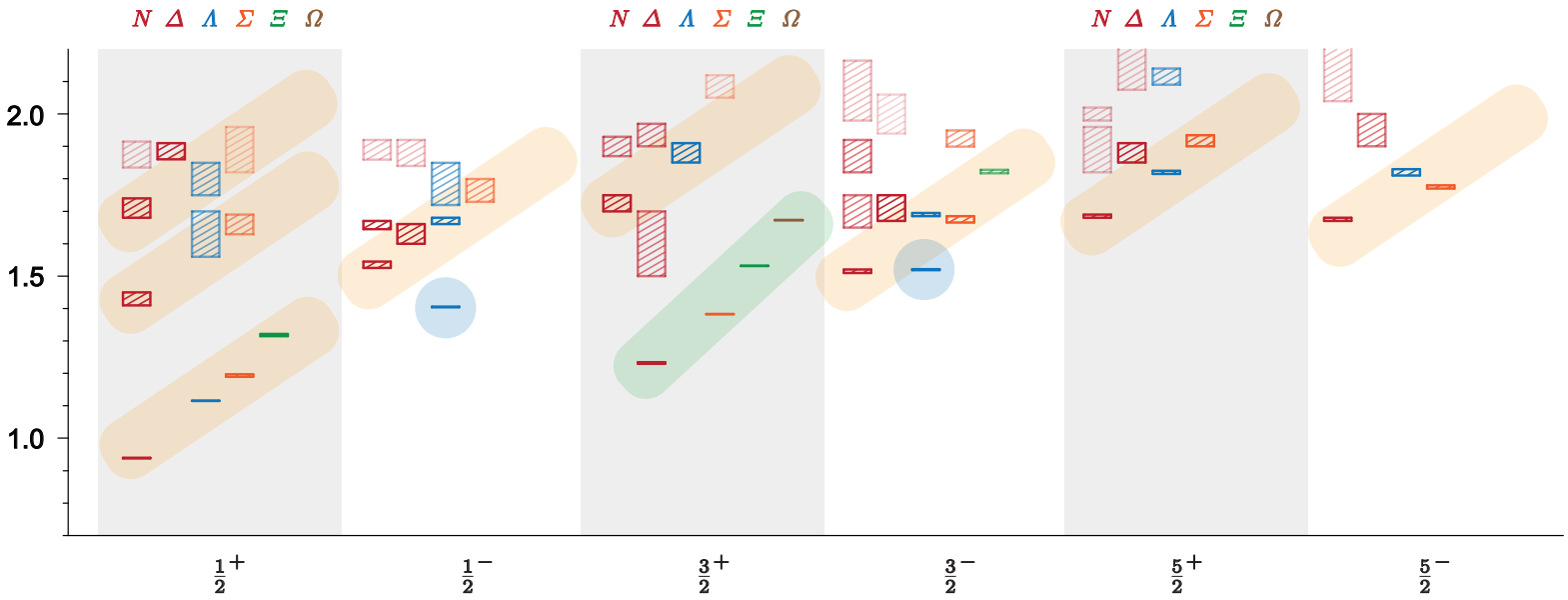
- states inside multiplet are not mass-degenerate
- Casimirs of $SU(3)_f$ are not good quantum numbers (these distinguish multiplets)
- But I_3, Y still conserved: states with same I_3, Y can mix
- In practice: isospin symmetry $SU(2)_f$ still good (u/d masses) states with same I, I_3, Y can mix

→ octet-decuplet mixing (Σ, Ξ)

→ singlet-octet mixing (Λ)



Dynamics?

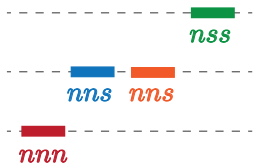


Expected patterns:

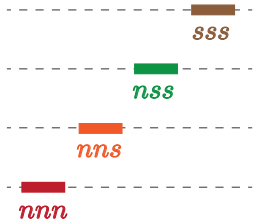
Singlet



Octet



Decuplet



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arXiv: 2503.10397 (for Encyclopedia of Particle Physics)

Burkert, GE, Klempt, "*The impact of* γN *and* γ*N *interactions on our understanding of nucleon excitations*", arXiv:2506.16482

GE, Sanchis-Alepuz, Williams, Alkofer, Fischer,
"*Baryons as relativistic three-quark bound states*",
Prog. Part. Nucl. Phys. 91 (2016), arXiv: 1606.09602

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"*Diquark correlations in hadron physics: Origin, impact and evidence*", Prog. Part. Nucl. Phys. 116 (2021),
arXiv: 2008.07630

Quark models

$$H\,\Psi = E\,\Psi \qquad H = H_{\rm kin} + \sum_{i<j} V(\mathbf{r}_{ij})$$



E.g. **Breit-Fermi interaction:**
non-relativistic limit of one-gluon exchange
de Rujula, Georgi, Glashow, PRD 12 (1975)

$$V(\mathbf{r}) \sim \alpha \left[-\frac{1}{r} + V_{\rm ss}(\mathbf{r}) + V_{\rm so}(\mathbf{r}) + \dots \right]$$

Coulomb **Spin-orbit**

Hyperfine (spin-spin) interaction:
generates splittings between $S = \frac{1}{2}, \frac{3}{2}$
$\Rightarrow$ e.g. N–Δ splitting

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de Rujula, Georgi, Glashow, PRD 12 (1975)
Isgur, Karl, PRD 20 (1979)
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Capstick, Isgur, PRD 34 (1986)

Goldstone-boson exchange
Glozman, Riska, Phys. Rept. 268 (1996)
Glozman, Plessas, Varga, Wagenbrunn, PRD 58 (1998)
Melde, Plessas, Sengl, PRD 77 (2008)

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Anselmino, Predazzi, Ekelin, Fredriksson, Lichtenberg, Rev. Mod. Phys. 65 (1993)
Ebert, Faustov, Galkin, PRD 72 (2005), PLB 659 (2008)
Santopinto, Ferretti, PRC 92 (2015), FBS 57 (2016)

String models
Bijker, Iachello, Leviatan, Annals Phys. 236 (1994), Annals Phys. 284 (2000)

Large-Nc expansion
Goity, Schat, Scoccola, PRD 66 (2002), PLB 564 (2003)
Matagne, Stancu, PRD 71 (2005), PRD 85 (2012)

Hypercentral quark model
Giannini, Santopinto, Vassallo, EPJ A 12 (2001)
Giannini, Santopinto, Chin. J. Phys. 53 (2015)

Light-front & holographic models
Brodsky, de Teramond, Dosch, Erlich, Phys. Rept. 584 (2015)

BSE models
Loring, Kretzschmar, Metsch, Petry, EPJ A 10 (2001)
Loring, Metsch, Petry, EPJ A 10 (2001)
Metsch, EPJ A 35 (2008)

. . .

Quark potential models

Total wave function:

$$\Psi = \Psi_{\text{dynamics}} \otimes \Psi_{\text{flavor}} \otimes \Psi_{\text{color}}$$

Color part is totally antisymmetric:

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10}_S \oplus \mathbf{8}_{M_A} \oplus \mathbf{8}_{M_S} \oplus \mathbf{1}_A$$

Singlet

Flavor part:

Must find **combined irreducible reps.** of permutation group S_3 (3 valence quarks) and $SU(N_f)$ (N_f flavors)

$$\overbrace{\phantom{\begin{pmatrix} u \\ d \\ s \\ \vdots \end{pmatrix} \otimes \begin{pmatrix} u \\ d \\ s \\ \vdots \end{pmatrix} \otimes \begin{pmatrix} u \\ d \\ s \\ \vdots \end{pmatrix}}}^{S_3}$$

$$\left.\begin{pmatrix} u \\ d \\ s \\ \vdots \end{pmatrix} \otimes \begin{pmatrix} u \\ d \\ s \\ \vdots \end{pmatrix} \otimes \begin{pmatrix} u \\ d \\ s \\ \vdots \end{pmatrix}\right\} SU(N_f)$$

$\Rightarrow$ Nucleon:

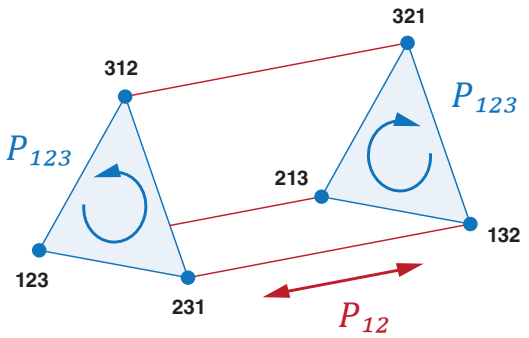
Proton: $\frac{1}{\sqrt{2}} \begin{bmatrix} udu - duu \\ -\frac{1}{\sqrt{3}}(udu + duu - 2uud) \end{bmatrix}$

Neutron: $\frac{1}{\sqrt{2}} \begin{bmatrix} udd - dud \\ \frac{1}{\sqrt{3}}(dud + udd - 2ddu) \end{bmatrix}$

Flavor

Permutation group S_3 consists of 6 group elements, can be reconstructed from **transposition** P_{12} ($1 \leftrightarrow 2$) and **cyclic permutation** P_{123} ($1 \to 2$, $2 \to 3$, $3 \to 1$):

$$1\ \psi_{123} = \psi_{123} \qquad P_{12}\ \psi_{123} = \psi_{213}$$
$$P_{123}\ \psi_{123} = \psi_{231} \qquad P_{12}\,P_{123}\ \psi_{123} = \psi_{132}$$
$$P_{123}^2\ \psi_{123} = \psi_{312} \qquad P_{12}\,P_{123}^2\ \psi_{123} = \psi_{321}$$

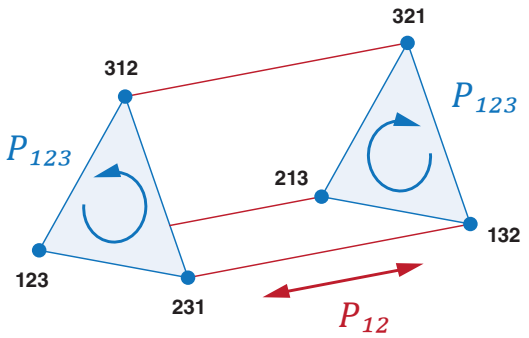


Cayley graph

Flavor

Permutation group S_3 consists of 6 group elements, can be reconstructed from **transposition** P_{12} $(1 \leftrightarrow 2)$ and **cyclic permutation** P_{123} $(1 \to 2,\ 2 \to 3,\ 3 \to 1)$:

$$1\ \psi_{123} = \psi_{123} \qquad P_{12}\ \psi_{123} = \psi_{213}$$
$$P_{123}\ \psi_{123} = \psi_{231} \qquad P_{12}\,P_{123}\ \psi_{123} = \psi_{132}$$
$$P_{123}^2\ \psi_{123} = \psi_{312} \qquad P_{12}\,P_{123}^2\ \psi_{123} = \psi_{321}$$



Cayley graph

Find combinations transforming under **irreps** of S_3 :

- **Singlet** $\mathcal{S}$ is invariant under any permutation:

$$P_{12}\,\mathcal{S} = \mathcal{S} \qquad \mathcal{S} = \psi_{123} + \psi_{231} + \psi_{312} + \psi_{213} + \psi_{132} + \psi_{321}$$
$$P_{123}\,\mathcal{S} = \mathcal{S}$$

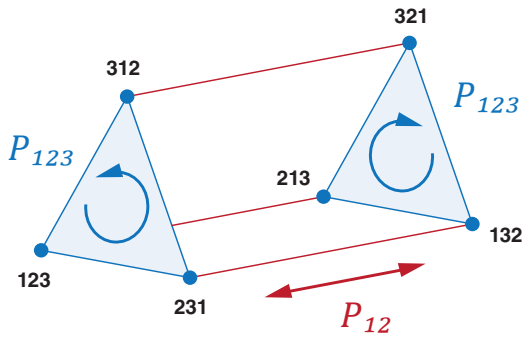
- **Antisinglet** $\mathcal{A}$ is antisymmetric under any transposition i $\leftrightarrow$ j:

$$P_{12}\,\mathcal{A} = -\mathcal{A} \qquad \mathcal{A} = \psi_{123} + \psi_{231} + \psi_{312} - \psi_{213} - \psi_{132} - \psi_{321}$$
$$P_{123}\,\mathcal{A} = \mathcal{A}$$

Flavor

Permutation group S_3 consists of 6 group elements, can be reconstructed from **transposition** P_{12} $(1 \leftrightarrow 2)$ and **cyclic permutation** P_{123} $(1 \to 2,\ 2 \to 3,\ 3 \to 1)$:

$$1\ \psi_{123} = \psi_{123} \qquad P_{12}\ \psi_{123} = \psi_{213}$$
$$P_{123}\ \psi_{123} = \psi_{231} \qquad P_{12}\,P_{123}\ \psi_{123} = \psi_{132}$$
$$P_{123}^2\ \psi_{123} = \psi_{312} \qquad P_{12}\,P_{123}^2\ \psi_{123} = \psi_{321}$$



Cayley graph

Find combinations transforming under **irreps** of S_3 :

- **Singlet** $\mathcal{S}$ is invariant under any permutation:

$$P_{12}\,\mathcal{S} = \mathcal{S} \qquad \mathcal{S} = \psi_{123} + \psi_{231} + \psi_{312} + \psi_{213} + \psi_{132} + \psi_{321}$$
$$P_{123}\,\mathcal{S} = \mathcal{S}$$

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$$P_{123}\,\mathcal{A} = \mathcal{A}$$

- **Doublets** $\mathcal{D}_1, \mathcal{D}_2$ form 2-dim subspace:

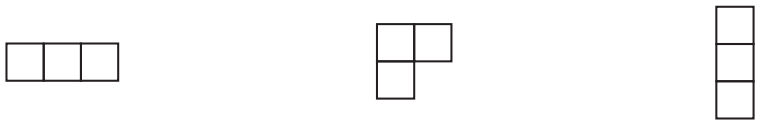
$$P_{12}\,\mathcal{D}_j = \mathbf{M}_{12}^{\mathrm{T}}\,\mathcal{D}_j \qquad \mathbf{M}_{12} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$P_{123}\,\mathcal{D}_j = \mathbf{M}_{123}^{\mathrm{T}}\,\mathcal{D}_j \qquad \mathbf{M}_{123} = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$$

$$\mathcal{D}_1 = \begin{bmatrix} (\psi_{231} - \psi_{132}) - (\psi_{312} - \psi_{321}) \\ \frac{1}{\sqrt{3}}\left((\psi_{231} + \psi_{132}) + (\psi_{312} + \psi_{321}) - 2(\psi_{123} + \psi_{213})\right) \end{bmatrix}$$

$$\mathcal{D}_2 = \begin{bmatrix} -\frac{1}{\sqrt{3}}\left((\psi_{231} - \psi_{132}) + (\psi_{312} - \psi_{321}) - 2(\psi_{123} - \psi_{213})\right) \\ (\psi_{231} + \psi_{132}) - (\psi_{312} + \psi_{321}) \end{bmatrix}$$

Flavor

Young diagrams:



$\mathcal{S}$ $\quad\quad$ $\mathcal{D}_1, \mathcal{D}_2$ $\quad\quad$ $\mathcal{A}$

Find combinations transforming under **irreps** of S_3 :

- **Singlet** $\mathcal{S}$ is invariant under any permutation:

$$P_{12}\,\mathcal{S} = \mathcal{S} \qquad P_{123}\,\mathcal{S} = \mathcal{S}$$

$$\mathcal{S} = \psi_{123} + \psi_{231} + \psi_{312} + \psi_{213} + \psi_{132} + \psi_{321}$$

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$$\mathcal{D}_1 = \begin{bmatrix} (\psi_{231} - \psi_{132}) - (\psi_{312} - \psi_{321}) \\ \frac{1}{\sqrt{3}}\left((\psi_{231} + \psi_{132}) + (\psi_{312} + \psi_{321}) - 2(\psi_{123} + \psi_{213})\right) \end{bmatrix}$$

$$\mathcal{D}_2 = \begin{bmatrix} -\frac{1}{\sqrt{3}}\left((\psi_{231} - \psi_{132}) + (\psi_{312} - \psi_{321}) - 2(\psi_{123} - \psi_{213})\right) \\ (\psi_{231} + \psi_{132}) - (\psi_{312} + \psi_{321}) \end{bmatrix}$$

Flavor

Now combine this with **SU(3),** e.g. for quark content uud:

$$\begin{aligned}
\psi_{123} &= u_1\, u_2\, d_3 = (uud)_{123} \\
\psi_{231} &= u_2\, u_3\, d_1 = (duu)_{123} \\
\psi_{312} &= u_3\, u_1\, d_2 = (udu)_{123} \\
\psi_{213} &= u_2\, u_1\, d_3 = (uud)_{123} \\
\psi_{132} &= u_1\, u_3\, d_2 = (udu)_{123} \\
\psi_{321} &= u_3\, u_2\, d_1 = (duu)_{123}
\end{aligned}
\quad \Longrightarrow \quad
\begin{aligned}
\mathcal{S} &= 2\,(uud + udu + duu) \quad \ldots\ \Delta^+ \\
\mathcal{A} &= 0
\end{aligned}$$

$$\mathcal{D}_1 = 2 \begin{bmatrix} duu - udu \\ \frac{1}{\sqrt{3}}\,(duu + udu - 2uud) \end{bmatrix} \quad \ldots \text{ proton}$$

$$\mathcal{D}_2 = 0$$

Find combinations transforming under **irreps** of S_3 :

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$$P_{12}\,\mathcal{S} = \mathcal{S} \qquad P_{123}\,\mathcal{S} = \mathcal{S}$$

$$\mathcal{S} = \psi_{123} + \psi_{231} + \psi_{312} + \psi_{213} + \psi_{132} + \psi_{321}$$

- **Antisinglet** $\mathcal{A}$ is antisymmetric under any transposition i $\leftrightarrow$ j:

$$P_{12}\,\mathcal{A} = -\mathcal{A} \qquad P_{123}\,\mathcal{A} = \mathcal{A}$$

$$\mathcal{A} = \psi_{123} + \psi_{231} + \psi_{312} - \psi_{213} - \psi_{132} - \psi_{321}$$

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$$\mathcal{D}_1 = \begin{bmatrix} (\psi_{231} - \psi_{132}) - (\psi_{312} - \psi_{321}) \\ \frac{1}{\sqrt{3}}\left((\psi_{231} + \psi_{132}) + (\psi_{312} + \psi_{321}) - 2(\psi_{123} + \psi_{213})\right) \end{bmatrix}$$

$$\mathcal{D}_2 = \begin{bmatrix} -\frac{1}{\sqrt{3}}\left((\psi_{231} - \psi_{132}) + (\psi_{312} - \psi_{321}) - 2(\psi_{123} - \psi_{213})\right) \\ (\psi_{231} + \psi_{132}) - (\psi_{312} + \psi_{321}) \end{bmatrix}$$

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\psi_{312} &= u_3\, u_1\, d_2 = (udu)_{123} \\
\psi_{213} &= u_2\, u_1\, d_3 = (uud)_{123} \\
\psi_{132} &= u_1\, u_3\, d_2 = (udu)_{123} \\
\psi_{321} &= u_3\, u_2\, d_1 = (duu)_{123}
\end{aligned}$$

$$\Rightarrow \quad \mathcal{S} = 2\,(uud + udu + duu) \quad \ldots\ \Delta^+$$

$$\mathcal{A} = 0$$

$$\mathcal{D}_1 = 2 \begin{bmatrix} duu - udu \\ \frac{1}{\sqrt{3}}\,(duu + udu - 2uud) \end{bmatrix} \quad \ldots \text{ proton}$$

$$\mathcal{D}_2 = 0$$

- ddu: exchange $u \leftrightarrow d$ $\Rightarrow$ $\mathcal{S} \ldots \Delta^0$, $\mathcal{D}_1 \ldots$ neutron
- uuu $\Rightarrow$ $\mathcal{S} \ldots \Delta^{++}$
- ddd $\Rightarrow$ $\mathcal{S} \ldots \Delta^-$
- uds $\Rightarrow$ $\mathcal{S} \ldots \Sigma^0$, $\mathcal{D}_1 \ldots \Sigma^0$, $\mathcal{D}_2 \ldots \Lambda^0$, $\mathcal{A} \ldots \Lambda^0$

$\Rightarrow$ SU(3) flavor wave functions:

	uuu	uud	ddu	ddd	uus	uds	dds	ssu	ssd	sss
$\mathcal{S}$	Δ^{++}	Δ^+	Δ^0	Δ^-	Σ^+	Σ^0	Σ^-	Ξ^0	Ξ^-	Ω^-
$\mathcal{D}_1$		p	n		Σ^+	Σ^0	Σ^-	Ξ^0	Ξ^-	
$\mathcal{D}_2$						Λ^0				
$\mathcal{A}$						Λ^0				

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} =$$

Decuplet $\quad \mathbf{10}_S \oplus$

Octet $\quad \mathbf{8}_{M_A} \oplus \mathbf{8}_{M_S} \oplus$

Singlet $\quad \mathbf{1}_A$

Flavor

		I	I_3	S		
uud	p	$1/2$	$1/2$	0	$\frac{1}{\sqrt{2}}\begin{bmatrix} udu-duu \\ -\frac{1}{\sqrt{3}}(udu+duu-2uud)\end{bmatrix}$	**octet**
udd	n	$1/2$	$-1/2$	0	$\frac{1}{\sqrt{2}}\begin{bmatrix} udd-dud \\ \frac{1}{\sqrt{3}}(dud+udd-2ddu)\end{bmatrix}$	
uus	Σ^+	1	1	-1	$\frac{1}{\sqrt{2}}\begin{bmatrix} usu-suu \\ -\frac{1}{\sqrt{3}}(usu+suu-2uus)\end{bmatrix}$	
uds	Σ^0	1	0	-1	$\frac{1}{2}\begin{bmatrix} sud-usd+sdu-dsu \\ \frac{1}{\sqrt{3}}(sud+usd+sdu+dsu-2uds-2dus)\end{bmatrix}$	
dds	Σ^-	1	-1	-1	$\frac{1}{\sqrt{2}}\begin{bmatrix} dsd-sdd \\ -\frac{1}{\sqrt{3}}(dsd+sdd-2dds)\end{bmatrix}$	
uds	Λ^0	0	0	-1	$\frac{1}{2}\begin{bmatrix} \frac{1}{\sqrt{3}}(2uds-2dus+usd-dsu+sdu-sud) \\ usd-dsu+sud-sdu\end{bmatrix}$	
uss	Ξ^0	$1/2$	$1/2$	-2	$\frac{1}{\sqrt{2}}\begin{bmatrix} uss-sus \\ \frac{1}{\sqrt{3}}(sus+uss-2ssu)\end{bmatrix}$	
dss	Ξ^-	$1/2$	$-1/2$	-2	$\frac{1}{\sqrt{2}}\begin{bmatrix} dss-sds \\ \frac{1}{\sqrt{3}}(sds+dss-2ssd)\end{bmatrix}$	
uuu	Δ^{++}	$3/2$	$3/2$	0	uuu	**decuplet**
uud	Δ^+	$3/2$	$1/2$	0	$\frac{1}{\sqrt{3}}(uud+udu+duu)$	
udd	Δ^0	$3/2$	$-1/2$	0	$\frac{1}{\sqrt{3}}(udd+dud+ddu)$	
ddd	Δ^-	$3/2$	$3/2$	0	ddd	
uus	Σ^+	1	1	-1	$\frac{1}{\sqrt{3}}(uus+usu+suu)$	
uds	Σ^0	1	0	-1	$\frac{1}{\sqrt{6}}(uds+sud+dsu+dus+usd+sdu)$	
dds	Σ^-	1	-1	-1	$\frac{1}{\sqrt{3}}(dds+dsd+sdd)$	
uss	Ξ^0	$1/2$	$1/2$	-2	$\frac{1}{\sqrt{3}}(uss+sus+ssu)$	
dss	Ξ^-	$1/2$	$-1/2$	-2	$\frac{1}{\sqrt{3}}(dss+sds+ssd)$	
sss	Ω^-	0	0	-3	sss	
uds	Λ^0	0	0	-1	$\frac{1}{\sqrt{6}}(uds+sud+dsu-dus-usd-sdu)$	**singlet**

$SU(3)_f$ is broken,
states with same I , I_3 , Y can mix

Flavor

Analogous for **SU(4):** $4 \otimes 4 \otimes 4 = \mathbf{20}_S \oplus \mathbf{20}_{M_A} \oplus \mathbf{20}_{M_S} \oplus \mathbf{4}_A$

	uuu	uud	ddu	ddd	uus	uds	dds	ssu	ssd	sss	uuc	udc	ddc	usc	dsc	ssc	ucc	dcc	scc	ccc
$\mathcal{S}$	Δ^{++}	Δ^{+}	Δ^{0}	Δ^{-}	Σ^{+}	Σ^{0}	Σ^{-}	Ξ^{0}	Ξ^{-}	Ω^{-}	Σ_c^{++}	Σ_c^{+}	Σ_c^{0}	Ξ_c^{+}	Ξ_c^{0}	Ω_c^{0}	Ξ_{cc}^{++}	Ξ_{cc}^{+}	Ω_{cc}^{+}	Ω_{ccc}^{+}
$\mathcal{D}_1$		p	n		Σ^{+}	Σ^{0}	Σ^{-}	Ξ^{0}	Ξ^{-}		Σ_c^{++}	Σ_c^{+}	Σ_c^{0}	Ξ_c^{+}	Ξ_c^{0}	Ω_c^{0}	Ξ_{cc}^{++}	Ξ_{cc}^{+}	Ω_{cc}^{+}	
$\mathcal{D}_2$						Λ^{0}						Λ_c^{+}		Ξ_c^{+}	Ξ_c^{0}					
$\mathcal{A}$						Λ^{0}						Λ_c^{+}		Ξ_c^{+}	Ξ_c^{0}					

single charm **double charm** **triple charm**

SU(2): $2 \otimes 2 \otimes 2 = 4_S \oplus 2_{M_A} \oplus 2_{M_S}$

	uuu	uud	ddu	ddd
$\mathcal{S}$	Δ^{++}	Δ^{+}	Δ^{0}	Δ^{-}
$\mathcal{D}_1$		p	n	
$\mathcal{D}_2$				
$\mathcal{A}$				

If this were **spin** instead of flavor:

Spin 3/2: $\uparrow\uparrow\uparrow$,
$\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow$,
$\uparrow\downarrow\downarrow + \downarrow\uparrow\downarrow + \downarrow\downarrow\uparrow$,
$\downarrow\downarrow\downarrow$

Spin 1/2: $\begin{bmatrix} \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow \\ -\frac{1}{\sqrt{3}}(\uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow - 2\uparrow\uparrow\downarrow) \end{bmatrix}$
$\begin{bmatrix} \uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow \\ \frac{1}{\sqrt{3}}(\uparrow\downarrow\downarrow + \downarrow\uparrow\downarrow - 2\downarrow\downarrow\uparrow) \end{bmatrix}$

Total wave function

$$\Psi_{\text{dynamics}} \otimes \Psi_{\text{flavor}} \otimes \Psi_{\text{color}} \overset{!}{=} \mathcal{A}$$

$$\{\mathcal{S}_f, \mathcal{D}_f, \mathcal{A}_f\} \qquad \mathcal{A}_c$$

suppose

$$= \Psi_{\text{orbital}} \otimes \Psi_{\text{spin}}$$

$$\{\mathcal{S}_o, \mathcal{D}_o, \mathcal{A}_o\} \quad \{\mathcal{S}_s, \mathcal{D}_s\}$$

$\Longrightarrow$ All **symmetric** group-theoretical combinations:

orb	spin-flavor	spin	flavor	
$\mathcal{S}_o$	$(\mathcal{D}_s \cdot \mathcal{D}_f)$	1/2	8	56-plet
$\mathcal{S}_o$	$(\mathcal{S}_s \mathcal{S}_f)$	3/2	10	(2·8 + 4·10)
$\mathcal{D}_o \cdot$	$(\mathcal{D}_s * \mathcal{D}_f)$	1/2	8	
$\mathcal{D}_o \cdot$	$(\mathcal{D}_s \mathcal{S}_f)$	1/2	10	70-plet
$\mathcal{D}_o \cdot$	$(\varepsilon \mathcal{D}_s)\, \mathcal{A}_f$	1/2	1	(2·8 + 2·10 + 2·1+ 4·8)
$\mathcal{D}_o \cdot$	$(\mathcal{S}_s \mathcal{D}_f)$	3/2	8	
$\mathcal{A}_o$	$(\mathcal{D}_s \wedge \mathcal{D}_f)$	1/2	8	20-plet
$\mathcal{A}_o$	$(\mathcal{S}_s \mathcal{A}_f)$	3/2	1	(2·8 + 4·1)

"SU(6)-symmetric quark model":

$$\mathbf{6} \otimes \mathbf{6} \otimes \mathbf{6} = \mathbf{56}_S \oplus \mathbf{70}_{M_A} \oplus \mathbf{70}_{M_S} \oplus \mathbf{20}_A$$

Total wave function

$$\Psi_{\text{dynamics}} \otimes \Psi_{\text{flavor}} \otimes \Psi_{\text{color}} \overset{!}{=} \mathcal{A}$$

$$\{\mathcal{S}_f, \mathcal{D}_f, \mathcal{A}_f\} \qquad \mathcal{A}_c$$

suppose

$$= \Psi_{\text{orbital}} \otimes \Psi_{\text{spin}}$$

$$\{\mathcal{S}_o, \mathcal{D}_o, \mathcal{A}_o\} \quad \{\mathcal{S}_s, \mathcal{D}_s\}$$

- Spin and flavor determined; orbital part from dynamics; **total angular momentum** $J = |L - S| \ldots L + S$; **parity** $P \implies J^P$
- **Orbital ground states:** $\mathcal{S}_o$, $L = 0$, $J = S$, $P = +$ $\implies$ octet with $J^P = \frac{1}{2}^+$, decuplet with $J^P = \frac{3}{2}^+$
- But: assumes that S and L are good quantum numbers (nonrelativistic quark model!)

$\implies$ All **symmetric** group-theoretical combinations:

orb	spin-flavor	spin	flavor	
$\mathcal{S}_o$	$(\mathcal{D}_s \cdot \mathcal{D}_f)$	½	8	56-plet
$\mathcal{S}_o$	$(\mathcal{S}_s\, \mathcal{S}_f)$	3/2	10	(2·8 + 4·10)
$\mathcal{D}_o \cdot (\mathcal{D}_s * \mathcal{D}_f)$		½	8	
$\mathcal{D}_o \cdot (\mathcal{D}_s\, \mathcal{S}_f)$		½	10	70-plet
$\mathcal{D}_o \cdot (\varepsilon \mathcal{D}_s)\, \mathcal{A}_f$		½	1	(2·8 + 2·10 + 2·1+ 4·8)
$\mathcal{D}_o \cdot (\mathcal{S}_s\, \mathcal{D}_f)$		3/2	8	
$\mathcal{A}_o$	$(\mathcal{D}_s \wedge \mathcal{D}_f)$	½	8	20-plet
$\mathcal{A}_o$	$(\mathcal{S}_s\, \mathcal{A}_f)$	3/2	1	(2·8 + 4·1)

"SU(6)-symmetric quark model":

$$\mathbf{6} \otimes \mathbf{6} \otimes \mathbf{6} = \mathbf{56}_S \oplus \mathbf{70}_{M_A} \oplus \mathbf{70}_{M_S} \oplus \mathbf{20}_A$$

Missing resonances (?)

Set up orbital wave functions in
spherical harmonic oscillator basis

Klempt, Richard, Rev. Mod. Phys. 82 (2010), Klempt, Metsch, EPJ A 48 (2012), Crede, Roberts, Rept. Prog. Phys. 76 (2013)

$$\phi_L(\boldsymbol{\rho},\boldsymbol{\lambda}) = \sum_{n_\rho,l_\rho,n_\lambda,l_\lambda} c^L_{n_\rho l_\rho n_\lambda l_\lambda} \left[\phi_{n_\rho l_\rho}(\boldsymbol{\rho}) \otimes \phi_{n_\lambda l_\lambda}(\boldsymbol{\lambda})\right]_L$$

Depend on Jacobi coordinates:

$$\boldsymbol{\rho} = \frac{\boldsymbol{x}_1 - \boldsymbol{x}_2}{\sqrt{2}}, \quad \boldsymbol{\lambda} = \frac{\boldsymbol{x}_1 + \boldsymbol{x}_2 - 2\boldsymbol{x}_3}{\sqrt{6}}$$

'Band quantum number' N with same energy:

$$N = 2n + l$$

$$n = n_\rho + n_\lambda \qquad n_\alpha > 0 \qquad \text{... radial excitations}$$
$$l = l_\rho + l_\lambda \qquad l_\alpha > 0 \qquad \text{... orbital excitations}$$
$$P = (-1)^l$$
$$L = |l_\rho - l_\lambda| \,...\, l_\rho + l_\lambda$$

Combining this with spin & flavor gives *lots* of states... **"Missing-resonances problem"**

N	L^P	O SF	S F	J^P
0	0^+	$\mathcal{S}_0$ **56**	$\frac{1}{2}$ **8**	$\frac{1}{2}^+$
			$\frac{3}{2}$ **10**	$\frac{3}{2}^+$
1	1^-	$\mathcal{D}_0$ **70**	$\frac{1}{2}$ **8**	$\frac{1}{2}^-, \frac{3}{2}^-$
			$\frac{1}{2}$ **10**	$\frac{1}{2}^-, \frac{3}{2}^-$
			$\frac{1}{2}$ **1**	$\frac{1}{2}^-, \frac{3}{2}^-$
			$\frac{3}{2}$ **8**	$\frac{1}{2}^-, \frac{3}{2}^-, \frac{5}{2}^-$
2	0^+	$\mathcal{S}_0$ **56**	$\frac{1}{2}$ **8**	$\frac{1}{2}^+$
			$\frac{3}{2}$ **10**	$\frac{3}{2}^+$
		$\mathcal{D}_0$ **70**	$\frac{1}{2}$ **8**	$\frac{1}{2}^+$
			$\frac{1}{2}$ **10**	$\frac{1}{2}^+$
			$\frac{1}{2}$ **1**	$\frac{1}{2}^+$
			$\frac{3}{2}$ **8**	$\frac{3}{2}^+$
	1^+	$\mathcal{A}_0$ **20**	$\frac{1}{2}$ **8**	$\frac{1}{2}^+, \frac{3}{2}^+$
			$\frac{3}{2}$ **1**	$\frac{1}{2}^+, \frac{3}{2}^+, \frac{5}{2}^+$
	2^+	$\mathcal{S}_0$ **56**	$\frac{1}{2}$ **8**	$\frac{3}{2}^+, \frac{5}{2}^+$
			$\frac{3}{2}$ **10**	$\frac{1}{2}^+, \frac{3}{2}^+, \frac{5}{2}^+, \frac{7}{2}^+$
		$\mathcal{D}_0$ **70**	$\frac{1}{2}$ **8**	$\frac{3}{2}^+, \frac{5}{2}^+$
			$\frac{1}{2}$ **10**	$\frac{3}{2}^+, \frac{5}{2}^+$
			$\frac{1}{2}$ **1**	$\frac{3}{2}^+, \frac{5}{2}^+$
			$\frac{3}{2}$ **8**	$\frac{1}{2}^+, \frac{3}{2}^+, \frac{5}{2}^+, \frac{7}{2}^+$

Bold: well-established states
Gray: no exp. candidate or 1 star

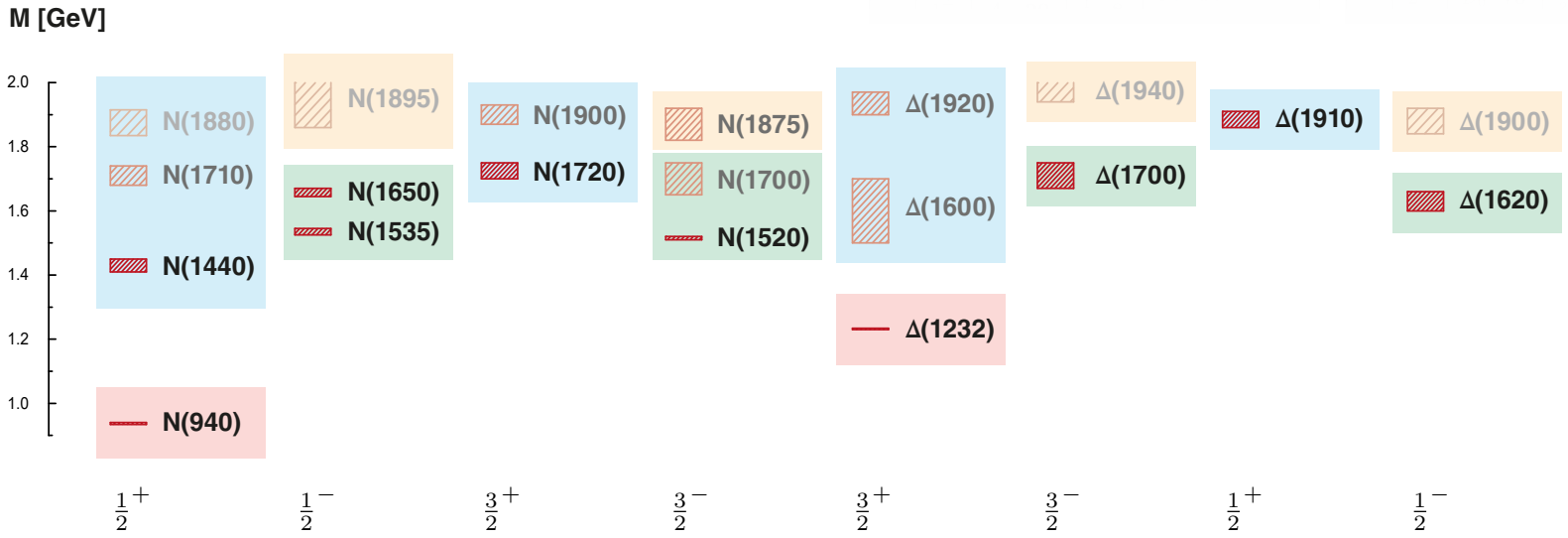
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3	1^-	$\mathcal{S}_0$ **56**	$\frac{1}{2}$ **8**	$\frac{1}{2}^-, \frac{3}{2}^-$
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		$\mathcal{D}_0$ **70**	$\frac{1}{2}$ **8**	$\frac{1}{2}^-, \frac{3}{2}^-$
			$\frac{1}{2}$ **10**	$\frac{1}{2}^-, \frac{3}{2}^-$
			$\frac{1}{2}$ **1**	$\frac{1}{2}^-, \frac{3}{2}^-$
			$\frac{3}{2}$ **8**	$\frac{1}{2}^-, \frac{3}{2}^-, \frac{5}{2}^-$
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		$\mathcal{D}_0$ **70**	$\frac{1}{2}$ **8**	$\frac{5}{2}^-, \frac{7}{2}^-$
			$\frac{1}{2}$ **10**	$\frac{5}{2}^-, \frac{7}{2}^-$
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			$\frac{3}{2}$ **1**	$\frac{3}{2}^-, \frac{5}{2}^-, \frac{7}{2}^-, \frac{9}{2}^-$

Missing resonances (?)

Prediction of many states, but also problems with **level ordering**, e.g. Roper vs. N(1535)

N	L^P	O SF	S F	J^P
0	0^+	$\mathcal{S}_0$ 56	$\frac{1}{2}$ 8	$\frac{1}{2}^+$
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2	0^+	$\mathcal{S}_0$ 56	$\frac{1}{2}$ 8	$\frac{1}{2}^+$
			$\frac{3}{2}$ 10	$\frac{3}{2}^+$
		$\mathcal{D}_0$ 70	$\frac{1}{2}$ 8	$\frac{1}{2}^+$
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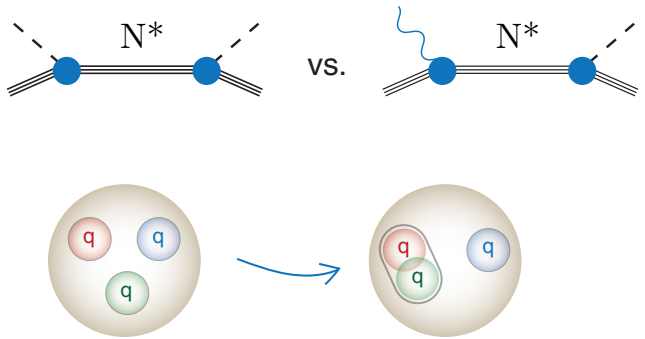
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			$\frac{1}{2}$ 10	$\frac{1}{2}^-, \frac{3}{2}^-$
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			$\frac{3}{2}$ 8	$\frac{1}{2}^-, \frac{3}{2}^-, \frac{5}{2}^-$
		$\mathcal{A}_0$ 20	$\frac{1}{2}$ 8	$\frac{1}{2}^-, \frac{3}{2}^-$
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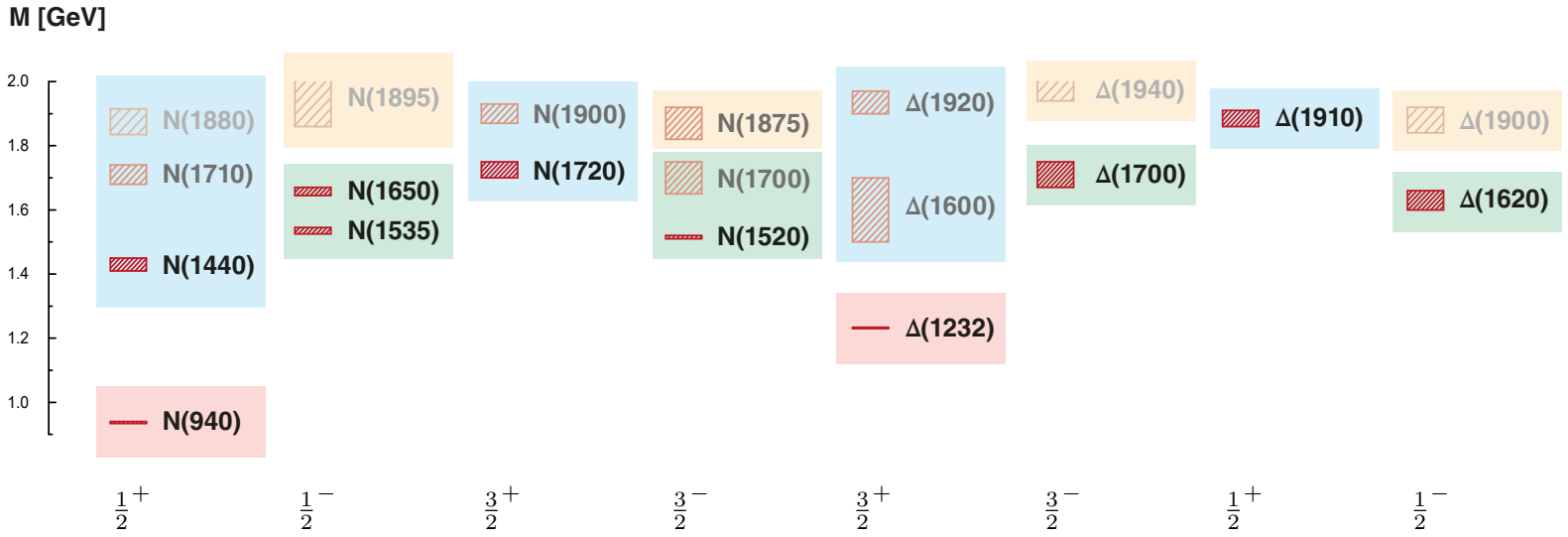


Missing resonances (?)

Prediction of many states, but also problems with **level ordering**, e.g. Roper vs. N(1535)

- We may not have found them yet (weak coupling to Nπ)
- **Diquark** clustering reduces # states?
- Or maybe our assumptions so far (non-relativistic quark model, harmonic oscillator) have been too restrictive?





1. Introduction
2. Spectrum & symmetries
3. Quark models
4. **QFT toolbox**
5. Functional methods
6. n-point functions & tensor bases
7. Multiquark states
8. Light-front wave functions

Lecture notes "QCD and hadron physics",
https://particle.uni-graz.at/en/quarks-hadrons-and-nuclei

GE, "*Hadron physics with functional methods*",
arXiv: 2503.10397 (for Encyclopedia of Particle Physics)

Burkert, GE, Klempt, "*The impact of* γN *and* γ*N *interactions on our understanding of nucleon excitations*", arXiv:2506.16482

GE, Sanchis-Alepuz, Williams, Alkofer, Fischer,
"*Baryons as relativistic three-quark bound states*",
Prog. Part. Nucl. Phys. 91 (2016), arXiv: 1606.09602

Barabanov, Bedolla, Brooks, Cates, Chen et al.,
"*Diquark correlations in hadron physics: Origin, impact and evidence*", Prog. Part. Nucl. Phys. 116 (2021),
arXiv: 2008.07630

Quantum Field Theory

$\mathbf{H}\,\Psi = \mathrm{E}\,\Psi \quad ?$

- Relativity
- No “wave function” with probability interpretation
- Particle creation & annihilation, Hilbert space becomes ∞ dim
- Renormalization
- . . .



Some issues addressed e.g. in Light-front Hamiltonian approaches

Brodsky, Teramond, Dosch, Erlich, Phys. Rept. 584 (2015)
Hiller, Prog. Part. Nucl. Phys. 90 (2016)

Typical baryon spectrum calculations in QFT do not start from Hamiltonian, except in certain limits or using approximations (non-relativistic, heavy-quark, ...)

... then how else?

Quantum Field Theory

- **Classical action:**

  $$S[\phi] = \int d^4x \, \mathcal{L}(\phi)$$

- **Relativity & unitarity:**

  $\phi_i(x)$ ... operators acting on states $| \, p_1 \ldots p_n \, \rangle$: vacuum $|0\rangle$, 1-particle states $|p\rangle$, etc.

  States transform under unitary representations of Poincaré group, probability interpretation for S-matrix elements $\langle \, p_1 \ldots p_n \, | \, q_1 \ldots q_n \, \rangle$

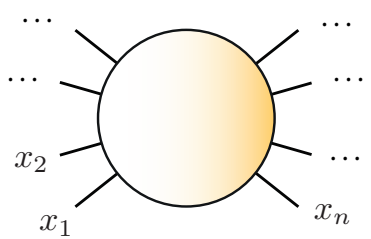
- **Causality:**

  Two measurements at spacelike distance cannot affect each other: $[\, O_1(x), \, O_2(y) \,] = 0$ for $(x-y)^2 < 0$

  $\Longrightarrow$ (anti) commutation relations for bosonic (fermionic) fields

- **Spectral condition**
- **Renormalizability** (for renormalizable theories)
- **Gauge invariance** (for gauge theories)

Interacting QFT: troubles at operator level,
but **correlation functions** mathematically well-defined

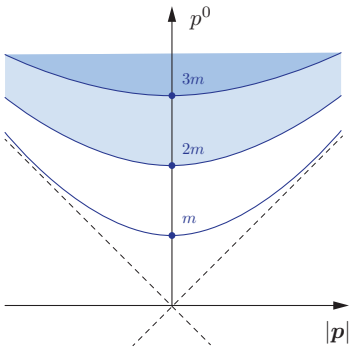


$$G(x_1, \ldots x_n) := \langle 0 | \mathsf{T} \, \phi(x_1) \ldots \phi(x_n) | 0 \rangle$$

- contain full content of QFT
- related to onshell S-matrix elements (LSZ formula)
- can be computed perturbatively (e.g. QED) but also **non-perturbatively**

Quantum Field Theory

Spectral condition



$|0\rangle$ ... **vacuum**

$|p\rangle$ ... **1-particle state** with onshell momentum $p^2 = m^2$, $\quad E_p^2 = \boldsymbol{p}^2 + m^2$

$|\, p_1 \ldots p_n \,\rangle$ ... **multiparticle states** with total momentum p and relative momenta, form continuum $p^2 = m_\lambda^2 > (2m)^2$

Completeness relation

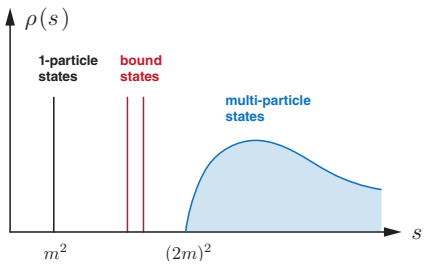
$$\mathbb{1} = \sum_\lambda \frac{1}{(2\pi)^3} \int \frac{d^3p}{2E_p} |\lambda\rangle\langle\lambda| \qquad p^2 = m_\lambda^2$$

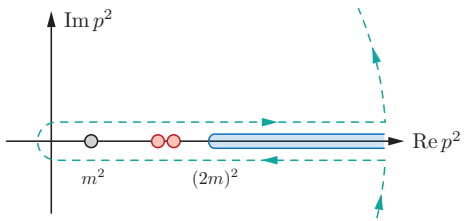
Inserting this in 2-point function yields **spectral representation:**

$$G(x,y) = \langle 0|\mathrm{T}\,\phi(x)\,\phi(y)|0\rangle$$

$$G(p^2) = \sum_\lambda \frac{iR_\lambda}{p^2 - m_\lambda^2 + i\epsilon} = \int_0^\infty \frac{ds}{2\pi} \frac{i\rho(s)}{p^2 - s + i\epsilon}$$

Spectral function





$\Rightarrow$ 1-particle states produce **poles** at $p^2 = m^2$

$\Rightarrow$ multiparticle states produce **branch cuts** for $p^2 > (2m)^2$

$\Rightarrow$ no singularities except on positive real axis (analyticity)

(what about gauge theories...?)

Quantum Field Theory

Correlation functions in **QCD:**

Two-point functions



quark propagator

gluon propagator

Three-point functions



quark-gluon vertex

three-gluon vertex

Higher n-point functions



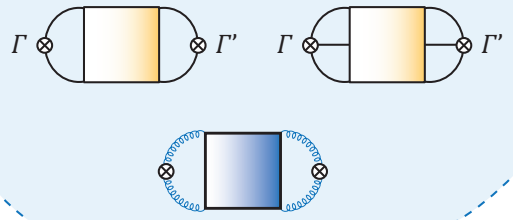
four-gluon vertex

quark 4-point function

quark 6-point function

Correlation functions with **composite** operators

$$\langle 0 | [\bar{\psi}\Gamma\psi](x)\,[\bar{\psi}\Gamma'\psi](y) | 0 \rangle$$



The information on each correlation function is contained in its (Lorentz-invariant) **dressing functions:**

$$G^{\mu\nu\ldots}_{\alpha\beta\ldots}(p_1,\ldots p_n) = \sum_{i=1}^{N} f_i(p_1^2, p_2^2, \ldots)\, \tau_i(p_1,\ldots p_n)^{\mu\nu\ldots}_{\alpha\beta\ldots}$$

Lorentz, Dirac, flavor, color

Quantum Field Theory

Hadrons produce poles in QCD's correlation functions:

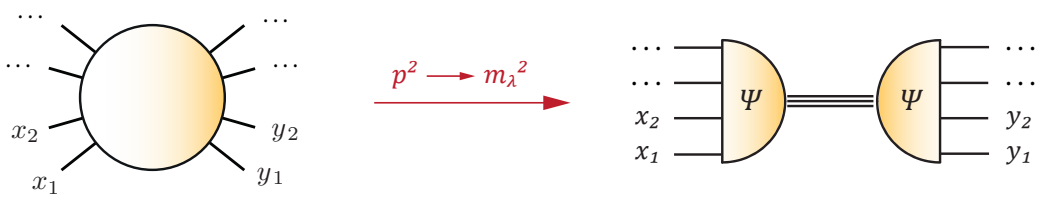
$$G(x_1, \dots x_r) = \langle 0 | \, \mathsf{T} \, \phi(x_1) \dots \phi(x_n) \, \phi(y_1) \dots \phi(y_r) \, | 0 \rangle$$

Proof: see e.g. books by
S. Weinberg, *The QFT of Fields* Vol. 1, 1995 (Sec. 10.2)
M. D. Schwartz, *QFT and the SM*, 2014 (Sec. 24.3)

Insert completeness relation with asymptotic, **onshell** states $|\lambda\rangle$:

$$\mathbb{1} = \sum_\lambda \frac{1}{(2\pi)^3} \int \frac{d^3p}{2E_p} \, |\lambda\rangle\langle\lambda| \qquad p^2 = m_\lambda^2 \;, \quad E_p^2 = \boldsymbol{p}^2 + m_\lambda^2$$

$$G(x_1, \dots x_r) = \int \frac{d^4p}{(2\pi)^4} \, e^{-ipz} \left[\frac{i \, \Psi\big(\{x_i\}, p\big) \, \Psi^\dagger\big(\{y_j\}, p\big)}{p^2 - m_\lambda^2 + i\epsilon} + \text{finite} \right]$$



Bethe-Salpeter wave function
encodes information about hadron

$$\Psi\left(\{x_i\}, p\right) = \langle 0 \, | \, \mathsf{T} \, \phi(x_1) \dots \phi(x_n) \, | \, \lambda \rangle$$

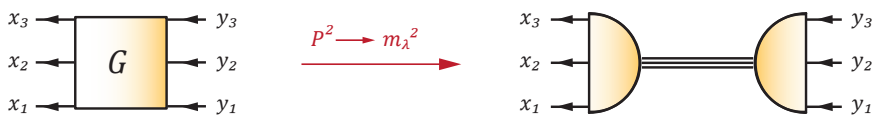
QFT analogue of wave function in QM
(but no probability interpretation)

- also holds for composite operators
- can insert 1 anywhere (and repeat)
- LSZ formula is special case
- poles in scattering amplitudes → poles in **experimental cross sections**

Quantum Field Theory

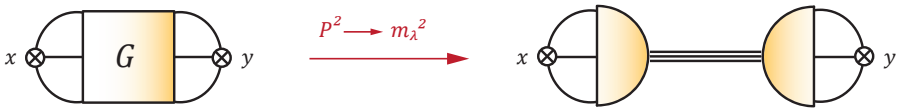
Baryon poles appear in any (elementary or composite) correlation function that is compatible with baryon quantum numbers:

- Elementary **quark 6-point function**:

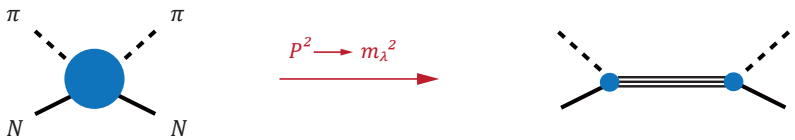


- Gauge-invariant (2-point) **current correlators:**

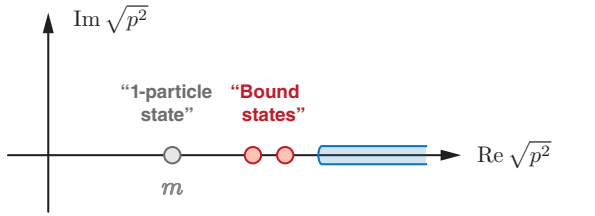
$$G(x-y) = \langle\, 0 \,|\, T\, \underbrace{[\Gamma_{\alpha\beta\gamma}\, \psi_\alpha\, \psi_\beta\, \psi_\gamma]\,(x)}_{B(x)}\, \underbrace{[\bar{\Gamma}_{\rho\sigma\tau}\, \bar{\psi}_\rho\, \bar{\psi}_\sigma\, \bar{\psi}_\tau]\,(y)}_{\bar{B}(y)} \,|\, 0 \,\rangle$$

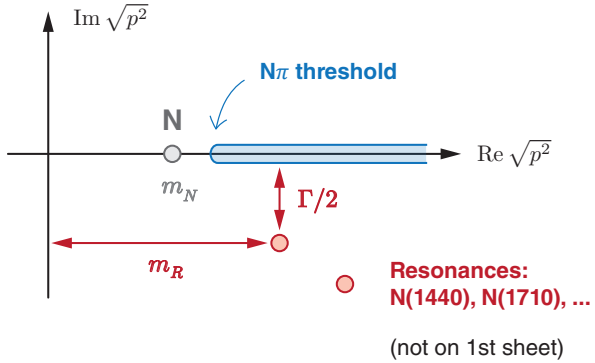


- **Hadronic scattering amplitudes:**



Singularity locations are independent of correlation function!





Effective field theories

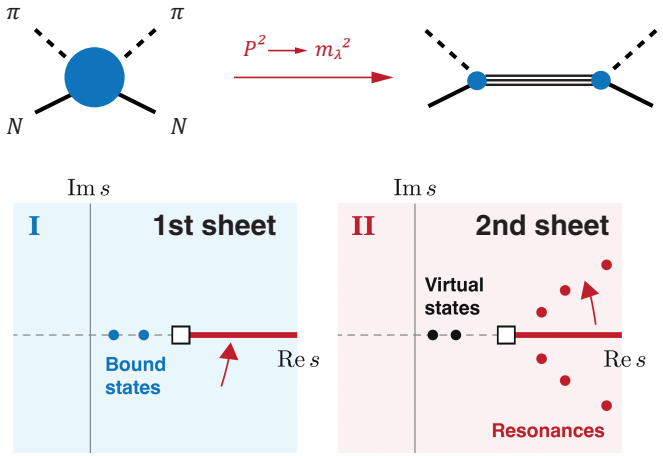
- **2 → 2 scattering amplitude**:
  Invariant matrix element

$$\mathcal{M}(s,z) = \sum_{l=0}^{\infty} (2l+1)\, f_l(s)\, P_l(z)$$

in CM frame:
$\sqrt{s}$ ... CM energy
z ... cosine of scattering angle

partial waves

Legendre polynomials



- **Unitarity** constrains imaginary parts of partial waves:

$$S^\dagger S = (1+iT)^\dagger (1+iT) \stackrel{!}{=} 1 \quad \Rightarrow \quad T - T^\dagger = i\, T^\dagger T \quad \Rightarrow$$

$$\mathrm{Im}\, f_l(s) = \tau(s)\, |f_l(s)|^2$$

$$\tau(s) = \frac{1}{16\pi} \sqrt{\frac{s-4m^2}{s}}$$

- Can be expressed through **phase shift**:

$$f_l(s) = R_l(s)\, e^{i\delta_l(s)} \quad \Rightarrow \quad R_l = \frac{1}{\tau} \sin\delta_l \quad \Rightarrow \quad f_l = \frac{1}{\tau}\, e^{i\delta_l} \sin\delta_l = \frac{1}{\tau}\, \frac{1}{\cot\delta_l - i} \quad \Rightarrow$$

$$\mathrm{Re}\, \frac{1}{f_l} = \tau\, \cot\delta_l\,, \qquad \mathrm{Im}\, \frac{1}{f_l} = -\tau$$

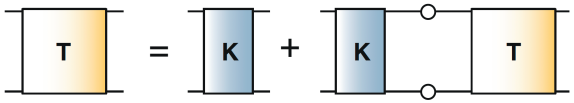
- Alternatively, through **K matrix**:

$$\mathrm{Re}\, \frac{1}{f_l} = \tau\, \cot\delta_l \; = \frac{1}{K_l} \quad \Rightarrow \quad f_l^{-1} = K_l^{-1} - i\tau \quad \Leftrightarrow \quad f_l = K_l + K_l\, i\tau\, f_l$$

Effective field theories

- **Bethe-Salpeter equation (BSE):**

$$\mathbf{T} = \mathbf{K} + \mathbf{K}\,\mathbf{G}_0\,\mathbf{T} \quad \Leftrightarrow \quad \mathbf{T}^{-1} = \mathbf{K}^{-1} - \mathbf{G}_0$$



T matrix: connected part of 4-point CF

2-particle irreducible **kernel**

free hadron **propagators**

symbolic notation: each loop stands for 4-momentum integral

- BSE automatically satisfies **unitarity**!

$$\underbrace{\mathbf{T}_+ - \mathbf{T}_-}_{\sim\, 2\,\mathrm{Im}\,\mathbf{T}} = \mathbf{T}_+(\mathbf{T}_-^{-1} - \mathbf{T}_+^{-1})\mathbf{T}_- = \mathbf{T}_+\underbrace{(\mathbf{G}_{0+} - \mathbf{G}_{0-})}_{\text{difference of propgators}}\mathbf{T}_- \quad \Rightarrow \quad \mathrm{Im}\, f_l(s) = \tau(s)\,|f_l(s)|^2$$

for partial waves

T matrix slightly above and below the cut

~ δ function

- BSE returns K-matrix equation:

$$\mathbf{G}_0 = \mathbf{G}_1 + \mathbf{G}_2 \quad \Rightarrow \quad \mathbf{T}^{-1} = \underbrace{\mathbf{K}^{-1} - \mathbf{G}_1}_{=:\,\mathbf{K}_1^{-1}} - \mathbf{G}_2 \quad \Rightarrow \quad \mathbf{T} = \mathbf{K}_1 + \mathbf{K}_1\,\mathbf{G}_2\,\mathbf{T}$$

if principle-value integral

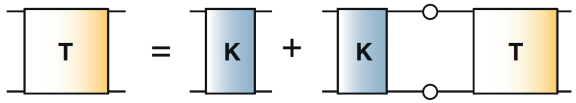
δ function

$$\Rightarrow \quad f_l = K_l + K_l\, i\tau\, f_l$$

Effective field theories

What is the kernel K?

- Particles in loops are hadrons
  ⇒ must be interactions between hadrons



- Derive from effective Lagrangians ⇒ **EFTs**
- Prime example is **chiral perturbation theory:** interactions between nucleons, pions, ...
  Weinberg, Physica A 96 (1979), Gasser, Leutwyler, Annals Phys. 158 (1984), Nucl. Phys. B 250 (1985)



Include all possible terms compatible with QCD symmetries (e.g., **chiral symmetry**)

→ systematic expansion in powers of derivatives & pion masses

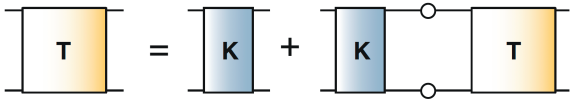
→ non-renormalizable, ∞ many terms with ∞ many free parameters,
but higher loops suppressed by derivative couplings ⇒ **low-energy EFT**,
introduce new **low-energy constants** (LECs) at each order

→ perturbative calculations of $M_N(m_\pi^2)$, $\pi\pi$ scattering, $N\pi$ scattering, ...

Reviews: Meißner, Rept. Prog. Phys. 56 (1993), Bernard, Kaiser, Meißner, Int. J. Mod. Phys. E 4 (1995),
Ecker, Prog. Part. Nucl. Phys. 35 (1995), Scherer, Adv. Nucl. Phys. 27 (2003),
Pich, Rept. Prog. Phys. 58 (1995), Bernard, Meißner, Ann. Rev. Nucl. Part. Sci. 57 (2007), ...

Effective field theories

But ChPT is perturbative ⇒ no **bound states & resonances** ⇒ **use BSE!** (“Unitarized ChPT”)



- BSE is **nonperturbative**! Like geometric series:

  $$f(x) = 1 + x + x^2 + x^3 + \ldots \quad \Rightarrow \quad f(x) = 1/(1-x) \quad \text{only if} \quad |x| < 1$$

  But $f(x) = 1 + xf(x) = 1 + x + x^2 f(x) = 1 + x + x^2 + x^3 f(x) = \ldots \quad \Leftrightarrow \quad f(x)^{-1} = 1 - x$ for any x

- BSEs with EFT kernels → **dynamical coupled channel equations**: $N\pi$, ΛK, ΣK, $N\gamma$, etc.
  ANL-Osaka, Jülich-Bonn-Washington, MAID, SAID, JLab, KSU, Giessen, Bonn-Gatchina, ...

  Mai, Meißner, Urbach, Phys. Rept. 1001 (2023), Guo et al., Rev. Mod. Phys. 90 (2018), Oller, Prog. Part. Nucl. Phys. 110 (2020), Döring, Haidenbauer, Mai, Sato, 2505.02745

- **Dynamically generated resonances / hadronic molecules** ↔ hadrons made of other hadrons.
  Many candidates: deuteron, light scalar mesons, heavy exotic mesons, $\Lambda(1405)$, Roper, ...

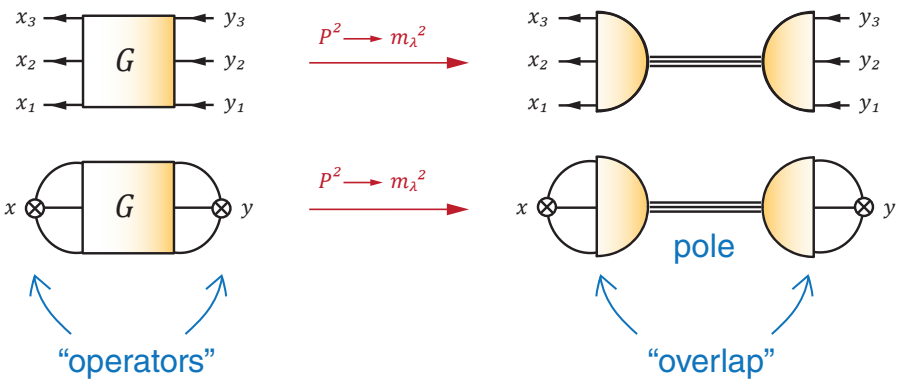
  Sometimes argued: if a state (e.g., baryon) is dynamically generated, it cannot be qqq state

  - But: a “state” is a pole in the scattering matrix, the same pole appears in qqq, $qqqq\bar{q}$, ...
    (unless forbidden by symmetry) → dynamically generated states **must** have qqq component
  - A BSE in EFT does not know about quarks, “all it can do” is generate resonances dynamically...
    need QCD calculations to decide!

Lattice QCD

- Calculate correlation functions directly from path integral:

$$G(x-y) = \langle 0 \,|\, T\, \underbrace{\left[\Gamma_{\alpha\beta\gamma}\, \psi_\alpha\, \psi_\beta\, \psi_\gamma\right](x)}_{B(x)}\, \underbrace{\left[\bar{\Gamma}_{\rho\sigma\tau}\, \bar{\psi}_\rho\, \bar{\psi}_\sigma\, \bar{\psi}_\tau\right](y)}_{\bar{B}(y)} \,|\, 0 \rangle \;=\; \int \mathcal{D}[\psi, \bar{\psi}, A]\, e^{-S[\psi,\bar{\psi},A]}\, B(x)\, \bar{B}(y)$$



Gauge-dependent n-point CF

↓

Gauge-invariant **two-point (current) correlator**

- **Nonperturbative, ab-initio**. Only need to supply current-quark masses & scale.
- Generate gauge-field ensembles, "measure" correlation functions on these ensembles, extract physical observables. Challenges: a > 0, L < ∞, physical pion masses (noisy)
- Extract **stable hadron masses** from correlator at large Euclidean times:

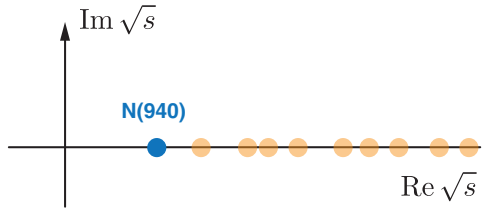
$$G(P) \xrightarrow{P^2 \to -m_\lambda^2} \frac{R_\lambda}{P^2 + m_\lambda^2} = \frac{R_\lambda}{P_4^2 + E_\lambda^2} \;, \qquad G(\tau, \boldsymbol{P}) \xrightarrow{P^2 \to -m_\lambda^2} R_\lambda \int \frac{dP_4}{2\pi} \frac{e^{iP_4\tau}}{P_4^2 + E_\lambda^2} = \frac{R_\lambda}{2E_\lambda} e^{-E_\lambda \tau} \;, \qquad E_\lambda^2 = \boldsymbol{P}^2 + m_\lambda^2$$

Lattice QCD

- Stable hadrons (N, Λ, Σ, Ξ, Ω) well established

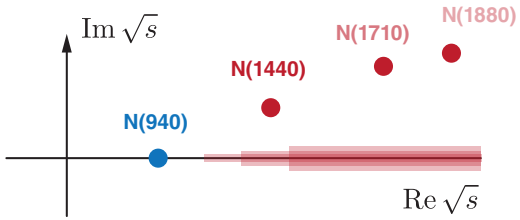


- **Unstable hadrons** above thresholds more difficult due to finite volume:



Finite volume:
energy levels
in finite box

Lüscher method

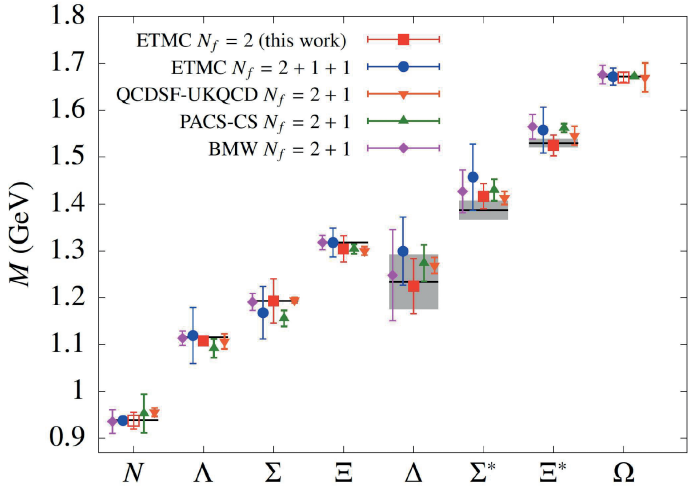


Infinite volume:
Branch cuts,
Riemann sheets,
resonance poles

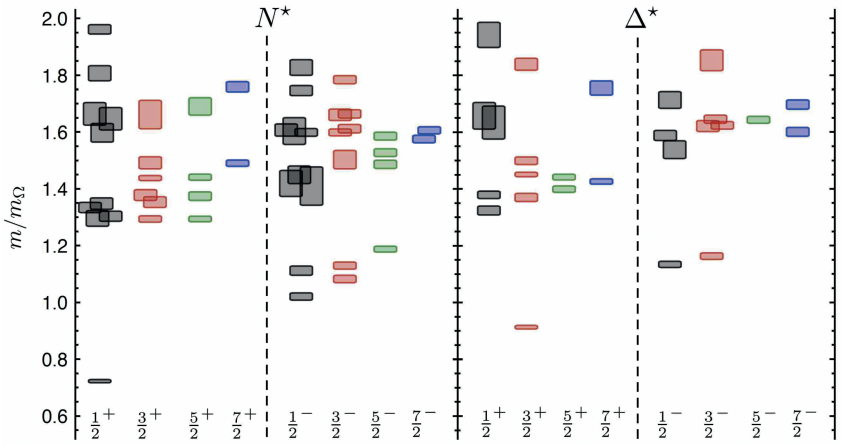
- **Step 1**: calculate energy levels in finite box, including qqq and multihadron operators

  (i.e., look for poles in qqq, $qqqq\bar{q}$, ... correlation matrix)

  Many calculations over past 15 years



Alexandrou, Kallidonis, PRD 96 (2017)



Edwards, Dudek, Richards, Wallace, PRD 84 (2011)

Lattice QCD

- **Step 2**: apply **Lüscher method**

  Lüscher, Commun. Math. Phys. 105 (1986), Nucl. Phys. B 354 (1991). **Reviews:** Briceno, Dudek, Young, Rev. Mod. Phys. 90 (2018), Hansen, Sharpe, Ann. Rev. Nucl. Part. Sci. 69 (2019), Mai, Döring, Rusetzky, EPJ ST 230 (2021)

  Can be formulated with BSE: lattice meets EFT

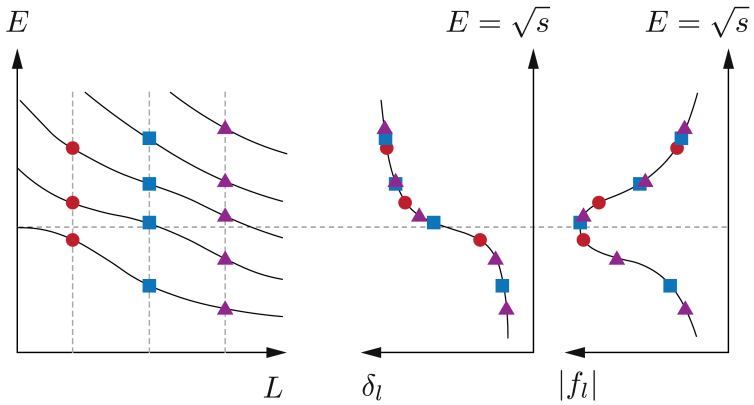
$$\mathbf{T}^{-1} = \mathbf{K}^{-1} - \mathbf{G}_0\,, \qquad \mathbf{T}_L^{-1} = \mathbf{K}_L^{-1} - \mathbf{G}_{0L}$$

  infinite volume: physical poles

  finite volume: poles ⇔ finite-volume energy levels

$$\mathbf{T}^{-1} - \mathbf{T}_L^{-1} = -(\mathbf{G}_0 - \mathbf{G}_{0L}) =: -\mathbf{F}$$ known function

$$\det\left(\mathbf{T} + \mathbf{F}^{-1}\right) = 0 \quad \text{at} \quad E = E_i(L)$$ Lüscher condition



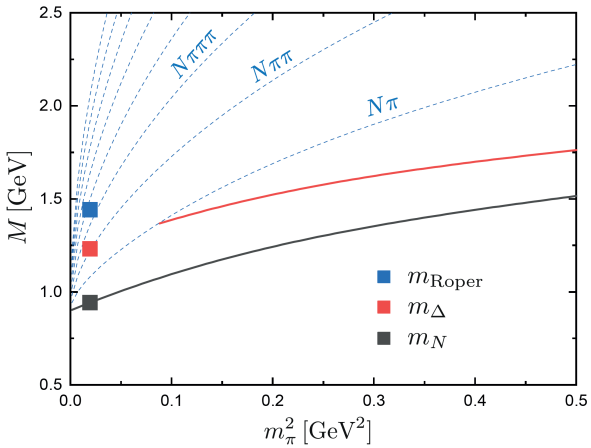
Each $E_i(L)$ maps to a value of **T**

- Much progress in meson sector. Also generalizations to states above **3-particle thresholds** (3-body BSE!)

- **Excited baryons**: first applications underway, but at low pion masses many thresholds open up → difficult

  e.g. Roper: no consensus yet, FV calculations differ, $N\pi\pi$ likely important, dynamically generated state?

  Lang, Leskovec, Padmanath, Prelovsek, PRD 95 (2017), Liu, IJMPE 26 (2017), Owa, Leinweber, Thomas, PRD 111 (2025)



1. Introduction
2. Spectrum & symmetries
3. Quark models
4. QFT toolbox
5. **Functional methods**
6. n-point functions & tensor bases
7. Multiquark states
8. Light-front wave functions

Lecture notes "QCD and hadron physics",
https://particle.uni-graz.at/en/quarks-hadrons-and-nuclei

GE, "*Hadron physics with functional methods*",
arXiv: 2503.10397 (for Encyclopedia of Particle Physics)

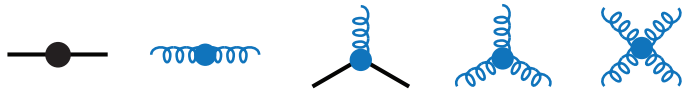
Burkert, GE, Klempt, "*The impact of* γN *and* γ*N *interactions on our understanding of nucleon excitations*", arXiv:2506.16482

GE, Sanchis-Alepuz, Williams, Alkofer, Fischer,
"*Baryons as relativistic three-quark bound states*",
Prog. Part. Nucl. Phys. 91 (2016), arXiv: 1606.09602

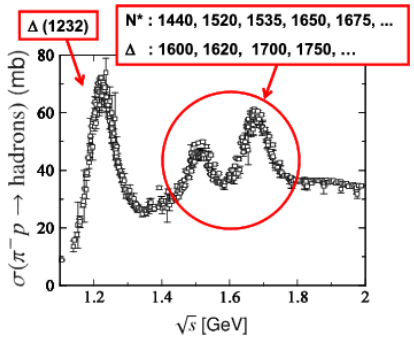
Barabanov, Bedolla, Brooks, Cates, Chen et al.,
"*Diquark correlations in hadron physics: Origin, impact and evidence*", Prog. Part. Nucl. Phys. 116 (2021),
arXiv: 2008.07630

Theory tools

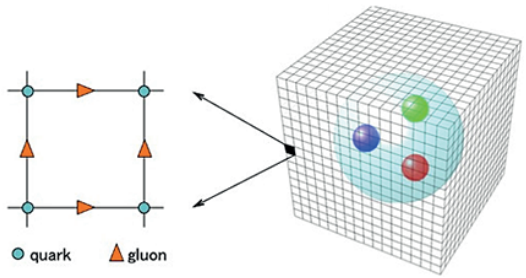
Functional methods
(Dyson-Schwinger & Bethe-Salpeter eqs, FRG, ...)



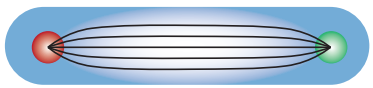
Amplitude analyses



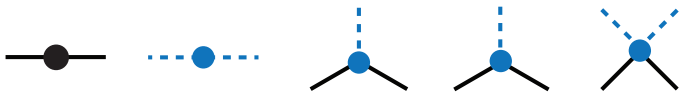
Lattice QCD



Phenomenological models
(Quark models, ...)

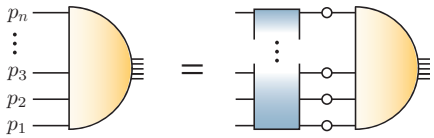


Effective theories
(Chiral perturbation theory, ...)



Functional methods

- Hadronic **bound-state equations** (Bethe-Salpeter & Faddeev eqs)

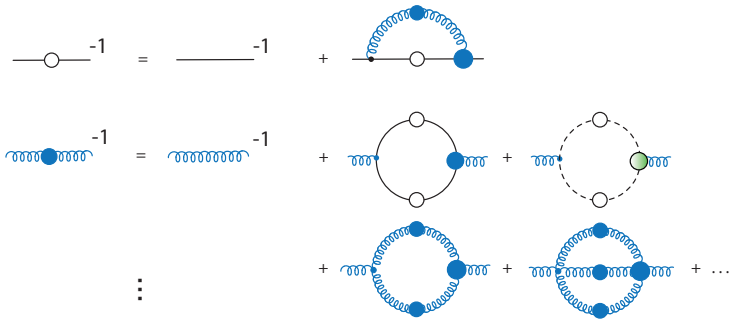


"QFT analogue of Schrödinger eq."

→ hadron masses & "wave functions"

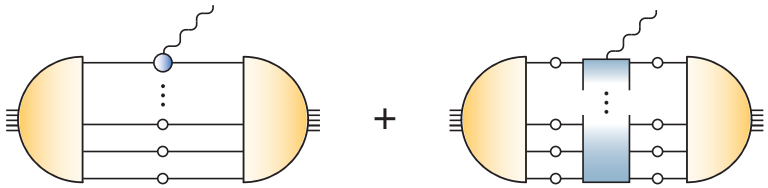
→ **spectroscopy calculations**

- Ingredients: **QCD's n-point functions,** Satisfy quantum eqs. of motion (DSEs)



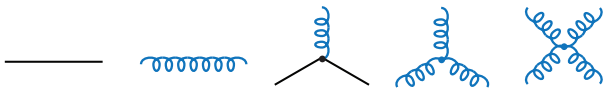
→ Dynamical mass generation, gluon mass gap, confinement, ...

- **Structure calculations:** form factors, PDFs, GPDs, TMDs, two-photon processes, ...

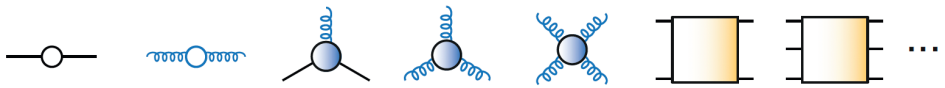


Derivation of DSEs

- **Classical action** $S[\phi]$ $\quad \phi \in \{\psi, \bar{\psi}, A, \dots\}$ **Classical eqs. of motion:** (Klein-Gordon, Dirac, Maxwell) $\dfrac{\delta S[\phi]}{\delta \phi(x)} = 0$



- **QFT: n-point correlation functions** $G(x_1, \dots x_n) = \langle 0 | \mathsf{T}\, \phi(x_1) \dots \phi(x_n) | 0 \rangle = \dfrac{\int \mathcal{D}\phi\, e^{iS[\phi]}\, \phi(x_1) \dots \phi(x_n)}{\int \mathcal{D}\phi\, e^{iS[\phi]}}$



Lattice: calculate them directly from path integral

Functional methods: calculate them *from each other* by relations derived from path integral

- How? Define **partition function:** $Z[J] = \int \mathcal{D}\phi\, e^{i\left(S[\phi] - \int d^4x\, \phi(x) J(x)\right)}$

$$\frac{i\delta}{\delta J(x_1)} \left[-i \int d^4x\, \phi(x) J(x) \right] = \phi(x_1)$$

$$\frac{i\delta}{\delta J(x_1)} Z[J] = \int \mathcal{D}\phi\, e^{i(\dots)}\, \phi(x_1)\,,$$

$$\frac{i\delta}{\delta J(x_1)} \frac{i\delta}{\delta J(x_2)} Z[J] = \int \mathcal{D}\phi\, e^{i(\dots)}\, \phi(x_1)\, \phi(x_2)$$

$$f\left[\frac{i\delta}{\delta J}\right] Z[J] = \int \mathcal{D}\phi\, e^{i(\dots)}\, f[\phi]$$

CFs are **functional derivatives** of Z[J]:

$$G(x_1 \dots x_n) = \frac{i\delta}{\delta J(x_1)} \cdots \frac{i\delta}{\delta J(x_n)} \bigg|_{J=0} \frac{Z[J]}{Z[0]}$$

From Z[J] one can get any CF; from all CFs one can reconstruct Z[J] ⇒ **contain full QFT**

Useful: define CFs also for J ≠ 0:

$$\langle f[\phi] \rangle_J := \frac{\int \mathcal{D}\phi\, e^{i\left(S[\phi] - \int d^4x\, \phi(x) J(x)\right)}\, f[\phi]}{\int \mathcal{D}\phi\, e^{i\left(S[\phi] - \int d^4x\, \phi(x) J(x)\right)}} = \frac{1}{Z[J]}\, f\left[\frac{i\delta}{\delta J}\right] Z[J]$$

Derivation of DSEs

- Z[J] is invariant under shift $\phi'(x) = \phi(x) + \varepsilon(x)$ (= relabeling of fields under integral):

$$
\begin{aligned}
Z[J] &= \int \mathcal{D}\phi'\, e^{i\left(S[\phi'] - \int d^4x\, \phi'(x) J(x)\right)} \\
&= \int \mathcal{D}\phi\, e^{i\left(S[\phi] - \int d^4x\, \phi(x) J(x)\right) + i \int d^4x\, \epsilon(x) \left(\frac{\delta S[\phi]}{\delta \phi(x)} - J(x)\right)} \\
&= Z[J] \left\langle e^{i \int d^4x\, \epsilon(x) \left(\frac{\delta S[\phi]}{\delta \phi(x)} - J(x)\right)} \right\rangle_J
\end{aligned}
$$

$$
\Rightarrow \quad \left\langle \frac{\delta S[\phi]}{\delta \phi(x)} \right\rangle_J = J(x)
$$

$$
\Rightarrow \quad \left\langle \frac{\delta S[\phi]}{\delta \phi(x)} \right\rangle = 0
$$

DSEs = ⟨ classical eqs. of motion ⟩
= **quantum eqs. of motion**

- To make this more useful, write:

$$
\frac{1}{Z[J]} \frac{\delta S}{\delta \phi}\left[\frac{i\delta}{\delta J}\right] Z[J] = J(x)
$$

... "master DSE"

functional derivatives of Z[J] generates CFs ⇒ **DSEs are relations between CFs**
(similar: classical symmetry relations → WTIs, STIs)

Useful: define CFs also for J ≠ 0:

$$
\langle f[\phi] \rangle_J := \frac{\int \mathcal{D}\phi\, e^{i\left(S[\phi] - \int d^4x\, \phi(x) J(x)\right)} f[\phi]}{\int \mathcal{D}\phi\, e^{i\left(S[\phi] - \int d^4x\, \phi(x) J(x)\right)}} = \frac{1}{Z[J]} f\left[\frac{i\delta}{\delta J}\right] Z[J]
$$

1PI correlation functions

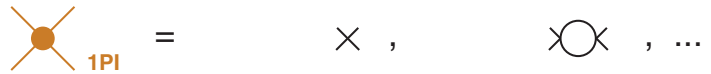
- Define **quantum effective action**:

$$Z[J] = \int \mathcal{D}\phi\, e^{i\left(S[\phi] - \int d^4x\, \phi(x)\, J(x)\right)} = e^{iW[J]} = e^{i\left(\Gamma[\varphi] - \int d^4x\, \varphi(x)\, J(x)\right)}$$

Schwinger functional — Effective action — (each of these encodes full QFT)

$Z[J] \xrightarrow{i\delta/\delta J}$ Full CFs:  (related to S-matrix elements)

$iW[J] \xrightarrow{i\delta/\delta J}$ Connected CFs:  (related to S-matrix elements)

$i\Gamma[\varphi] \xrightarrow{\delta/\delta\varphi}$ 1PI (1-particle-irreducible) CFs:  (only these are needed)

- $\varphi(x)$ is the **quantum averaged field:**

$$\varphi(x) = -\frac{\delta W[J]}{\delta J(x)} = \frac{1}{Z[J]}\frac{i\delta}{\delta J(x)} Z[J] \quad = \quad \langle \phi(x) \rangle_J \xrightarrow{J=0} 0, \qquad J(x) = \frac{\delta \Gamma[\varphi]}{\delta \varphi(x)}$$

- Then the CF for J ≠ 0 turns into

after some calculations :) — full propagator

$$\langle f[\phi] \rangle_J = \frac{\int \mathcal{D}\phi\, e^{i\left(S[\phi] - \int d^4x\, \phi(x) J(x)\right)} f[\phi]}{\int \mathcal{D}\phi\, e^{i\left(S[\phi] - \int d^4x\, \phi(x) J(x)\right)}} = \frac{1}{Z[J]} f\left[\frac{i\delta}{\delta J}\right] Z[J] \quad = \quad f\left[\varphi(x) + \int d^4y\, D_{xy}[\varphi]\, \frac{\delta}{\delta \varphi(y)}\right] 1$$

1PI correlation functions

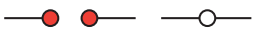
full propagator

$$\langle f[\phi]\rangle_J = f\left[\varphi(x) + \int d^4y\, D_{xy}[\varphi]\,\frac{\delta}{\delta\varphi(y)}\right] 1$$

- Can be used to relate **full** and **1PI** CFs:

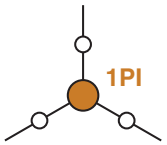
$$\langle\phi(x)\rangle_J = \left(\varphi(x) + \int d^4y\, D_{xy}[\varphi]\,\frac{\delta}{\delta\varphi(y)}\right) 1 = \varphi(x) \xrightarrow{\varphi=0} 0$$

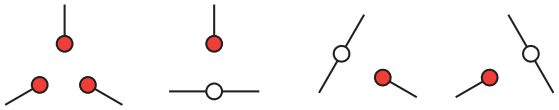
$$\langle\phi(x)\,\phi(y)\rangle_J = \left(\varphi(x) + \int d^4a\, D_{xa}\,\frac{\delta}{\delta\varphi_a}\right)\left(\varphi(y) + \int d^4b\, D_{yb}\,\frac{\delta}{\delta\varphi_b}\right) 1 = \varphi(x)\,\varphi(y) + D_{xy} \xrightarrow{\varphi=0}$$

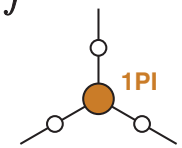


$$\langle\phi(x)\,\phi(y)\,\phi(z)\rangle_J = \left(\varphi(x) + \int d^4a\, D_{xa}\,\frac{\delta}{\delta\varphi_a}\right)(\varphi(y)\,\varphi(z) + D_{yz})$$

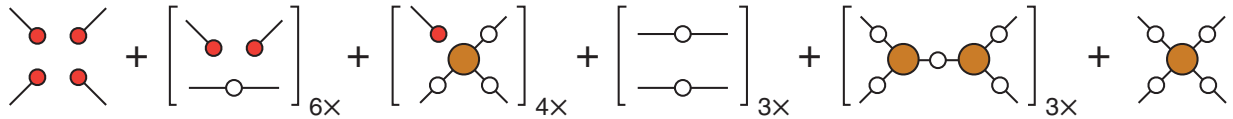
$$= \varphi_x\,\varphi_y\,\varphi_z + D_{xy}\,\varphi_z + D_{xz}\,\varphi_y + D_{yz}\,\varphi_x + \int d^4a \int d^4b \int d^4c\, D_{xa}\, D_{yb}\, D_{zc}\, U_{abc} \xrightarrow{\varphi=0}$$







$$\langle\phi(x)\,\phi(y)\,\phi(z)\,\phi(w)\rangle_J =$$



General: $\langle\phi^n\rangle = \left(\varphi + D\,\frac{\delta}{\delta\varphi}\right)\langle\phi^{n-1}\rangle, \qquad \frac{\delta}{\delta\varphi}\,(\underbrace{-D^{-1}}_{\text{1PI CFs}}, U, V, \ldots) = (U, V, \ldots)$

1PI correlation functions

full propagator

$$\langle f[\phi]\rangle_J = f\left[\varphi(x) + \int d^4y\, \boldsymbol{D_{xy}[\varphi]}\, \frac{\delta}{\delta\varphi(y)}\right] 1$$

- But can also be read like this:

$$f[\phi] \longrightarrow \langle f[\phi]\rangle_J = f\left(\varphi_x + \int d^4y\, D_{xy}\, \frac{\delta}{\delta\varphi_y}\right) = f(\varphi) + (\dots)$$

classical relation between fields

classical quantum



"connect the dots" in all possible ways!

- **Dyson-Schwinger equations:**

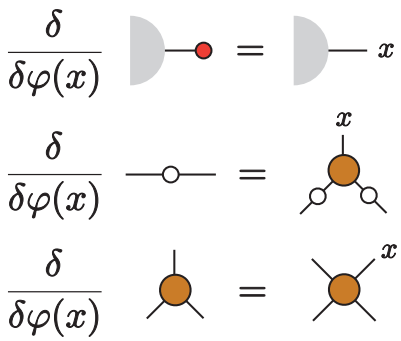
$$J(x) = \left\langle \frac{\delta S}{\delta\phi(x)} \right\rangle_J$$ ... master DSE

$$\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} = \frac{\delta S}{\delta\phi}\left[\varphi(x) + \int d^4y\, D_{xy}[\varphi]\, \frac{\delta}{\delta\varphi(y)}\right] = \underbrace{\frac{\delta S[\varphi]}{\delta\varphi(x)}}_{\text{classical}} + \underbrace{(\dots)}_{\text{quantum}}$$

functional derivatives

$$\left.\frac{\delta^n\Gamma}{\delta\varphi^n}\right|_{\varphi=0} = \dots$$ DSEs for 1PI CFs

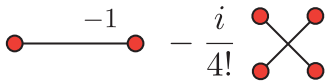
- **Graphical rules:**



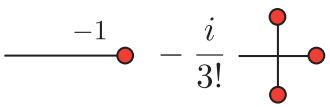
Derivation of DSEs: ϕ^4 theory

- Classical action:

$$S = \int d^4x \left[\frac{1}{2} \left(\partial^\mu \phi \, \partial_\mu \phi - m^2 \phi^2 \right) - \frac{g}{4!} \phi^4 \right] = \int d^4x \left[-\frac{1}{2} \phi \left(\Box + m^2 \right) \phi - \frac{g}{4!} \phi^4 \right] =$$



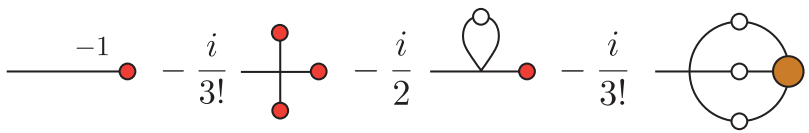
$$\frac{\delta S}{\delta \varphi} = -(\Box + m^2)\,\phi - \frac{g}{3!}\,\phi^3 \; = \; i$$



(= 0 ⇒ **classical eqs. of motion**)

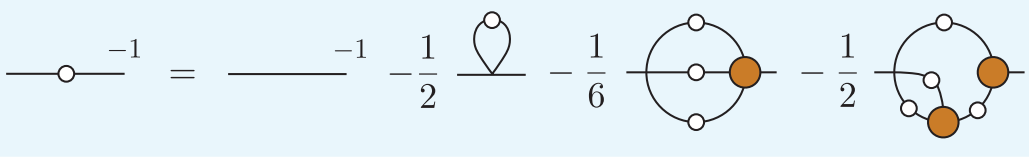
- **Quantum eqs. of motion** = master DSE: (connect dots in all possible ways)

$$\frac{\delta \Gamma}{\delta \varphi} = i$$



- Another functional derivative gives **DSE for full inverse propagator**:

$$-i \left. \frac{\delta^2 \Gamma}{\delta \varphi^2} \right|_{\varphi=0} =$$

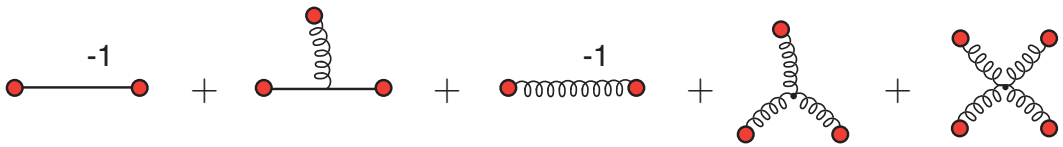


depends on full
3- and 4-point vertices
→ satisfy own DSEs

- DSE = **nonperturbative integral equation**;
  if we knew exact 3- and 4-point vertices, would get exact propagator.
  If coupling g is small ⇒ recovers perturbation theory

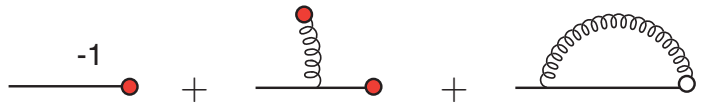
$$D^{-1} = D_0^{-1} - \Sigma \quad \Leftrightarrow \quad D = D_0 + D_0 \, \Sigma \, D = D_0 + D_0 \, \Sigma \, D_0 + D_0 \, \Sigma \, D_0 \, \Sigma \, D_0 + \ldots$$

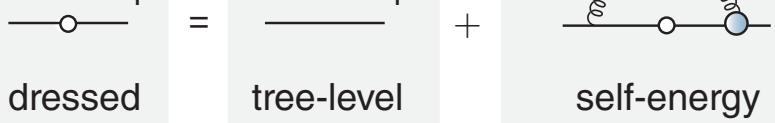
Derivation of DSEs: QCD

- Classical action: S = 

- Classical EoM (Dirac equation) for **quark:** $\frac{\delta S}{\delta \bar{\psi}}$ = 

= 0

- Quantum EoM for **quark:** $\frac{\delta \Gamma}{\delta \bar{\psi}}$ = 

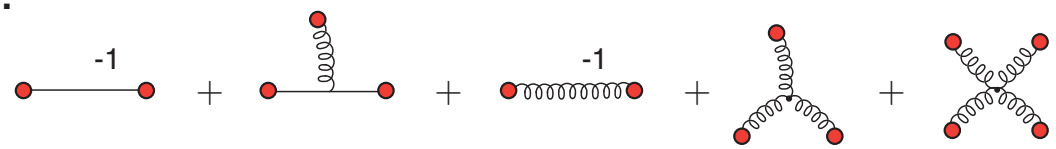
- **Quark DSE:** $\left.\frac{\delta^2 \Gamma}{\delta \bar{\psi} \delta \psi}\right|_{A,\psi,\bar{\psi}=0}$ = 

dressed = tree-level + self-energy

Easy!

Derivation of DSEs: QCD

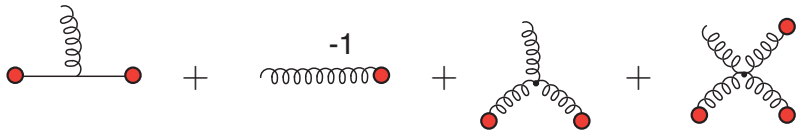
Classical action:

S = 

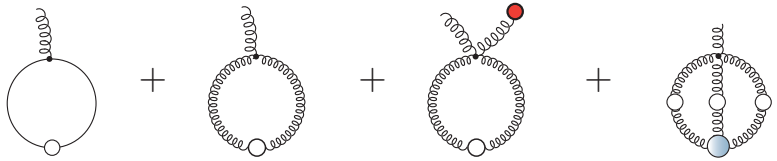
(ignoring prefactors & i's)

(ignoring gauge fixing ⇒ ghosts)

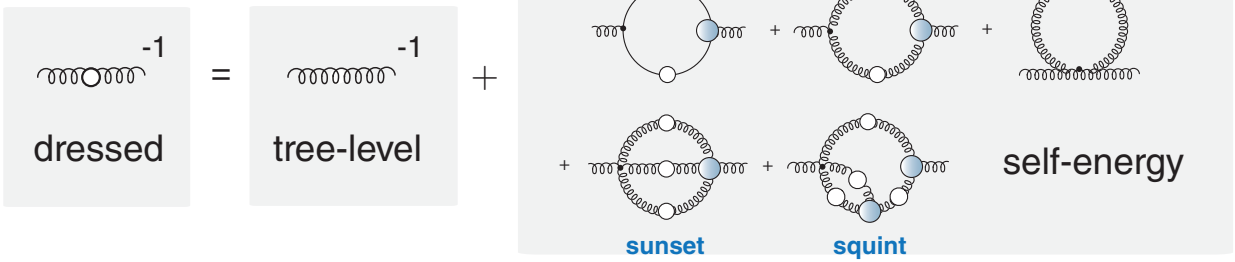
Gluon: classical equation of motion (Maxwell equation)

$\frac{\delta S}{\delta A}$ =  = 0

Gluon: quantum equation of motion

$\frac{\delta \Gamma}{\delta A}$ = (— ″ —) + 

Gluon DSE:

$\left.\frac{\delta^2 \Gamma}{\delta A^2}\right|_{A,\psi,\bar{\psi}=0}$ = 

DSEs

- **Quantum equations of motion**: exact equations in QFT

  Roberts, Williams, Prog. Part. Nucl. Phys. 33 (1994), Alkofer, Smekal, Phys. Rept. 353 (2001), ...

- Like Feynman diagrams, but ingredients are dressed ⇒ **integral equations**

- **Nonperturbative**, return perturbation theory if coupling is small

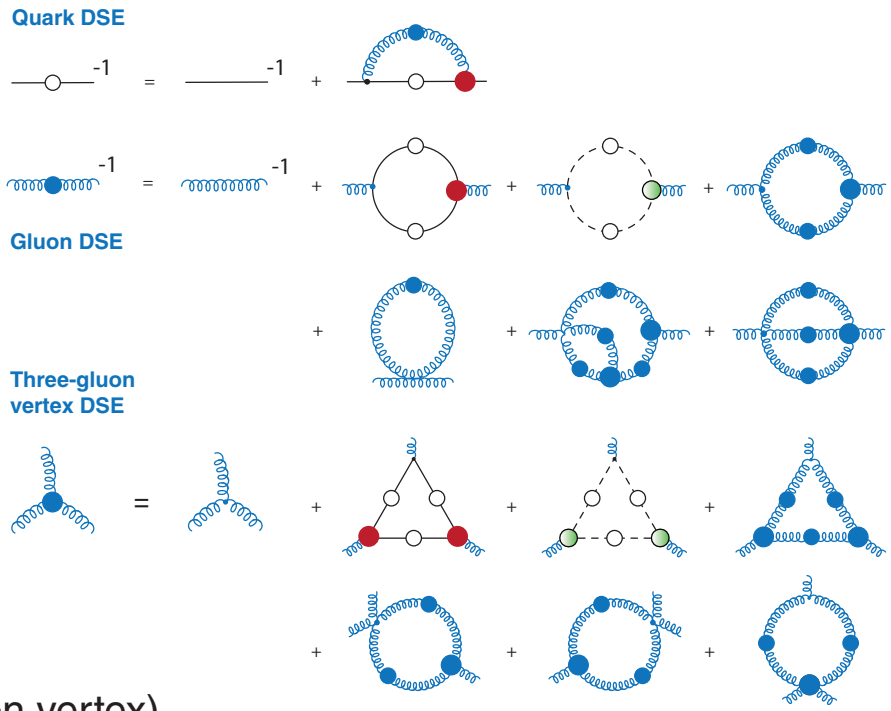
- Set of all n-point CFs determines QFT

- Each DSE contains higher n-point CFs ⇒ **infinitely coupled system**

- But do we need to know *every* CF (e.g., 127-gluon vertex) to make predictions? Higher n-point CFs are suppressed by momentum powers, higher loops by propagators

  ⇒ **truncations:** solve **closed subset** of equations by neglecting higher n-point CFs or using ansätze

  ⇒ **systematically improvable**: can always enlarge system to get closer to full tower of exact equations

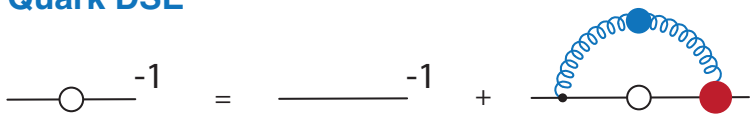


- Similar: **functional renormalization group**

  Berges, Tetradis, Wetterich, Phys. Rept. 363 (2002), Pawlowski, Annals. Phys. 322 (2007), Dupuis et al., Phys. Rept. 910 (2021)

Quark DSE

Quark DSE



- Quark propagator = two-point function, basic object describing a quark in QCD:

$$S_{\alpha\beta}(p) = \int d^4x\, e^{-ipx} \langle 0 | \mathrm{T}\, \psi_\alpha(x)\, \bar{\psi}_\beta(0) | 0 \rangle$$

- In momentum space: 2 dressing functions

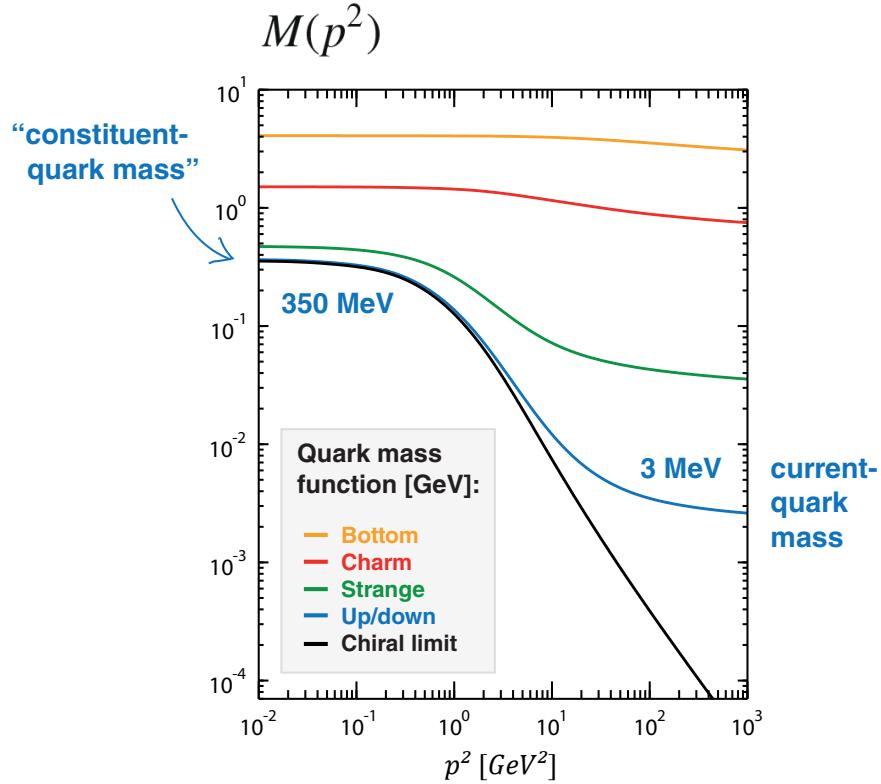
quark mass function

$$S(p)^{-1} = A(p^2)\left(i\not{p} + M(p^2)\right) \quad \Leftrightarrow$$

$$S(p) = \frac{1}{A(p^2)} \frac{-i\not{p} + M(p^2)}{p^2 + M(p^2)^2} = -i\not{p}\,\sigma_v(p^2) + \sigma_s(p^2)$$

- Compare: propagator for free spin-1/2 particle

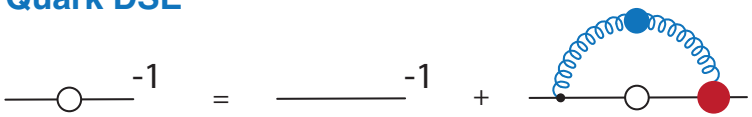
$$S(p) = \frac{-i\not{p} + m}{p^2 + m^2} \quad \Leftrightarrow \quad \sigma_v(p^2) = \frac{1}{p^2 + m^2}, \quad \sigma_s(p^2) = \frac{m}{p^2 + m^2}$$

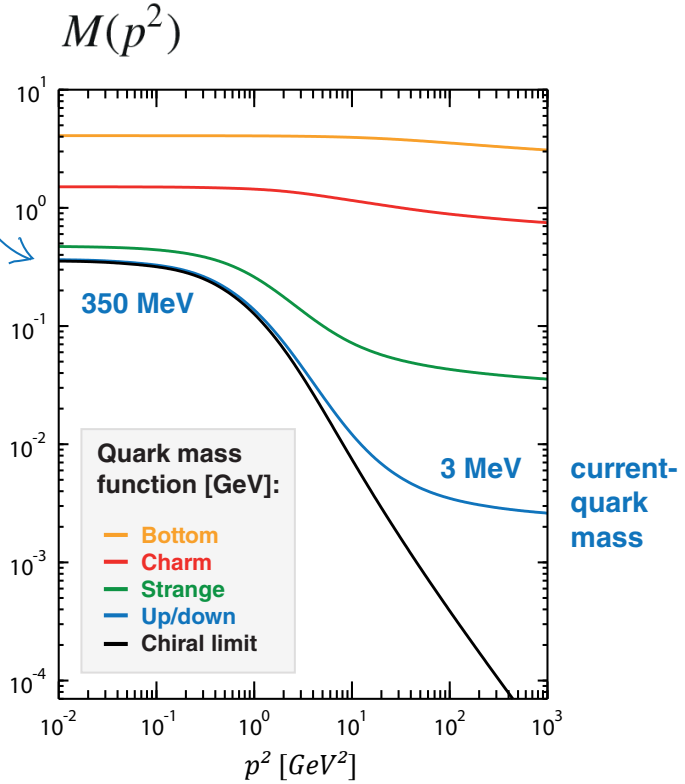


- Apparently, something drastic must be happening here: **dynamical mass generation**

Quark DSE

Quark DSE





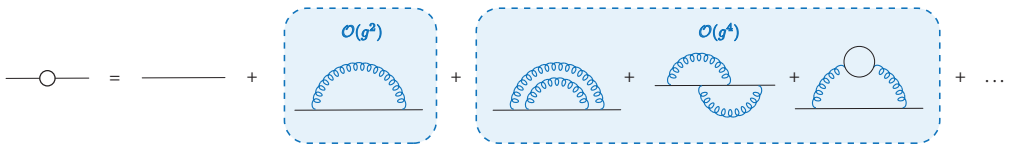
- Remember **chiral symmetry**:

$$SU(N_f)_V: \quad \psi' = e^{i\sum_a \varepsilon_a t_a}\psi\,, \qquad SU(N_f)_A: \quad \psi' = e^{i\gamma_5\sum_a \varepsilon_a t_a}\psi$$

- If chiral symmetry in classical Lagrangian were preserved in QFT, all n-point CFs would be chirally symmetric

$$\{\gamma_5, S(p)\} = 0 \quad \Rightarrow \quad M(p^2) = 0$$

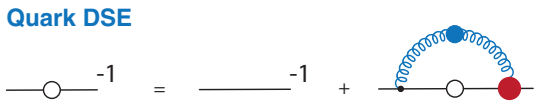
- This is true in perturbation theory: each diagram has odd # gamma matrices ⇒ Trace = 0 ⇒ M(p²) = 0



- So it must be a **nonperturbative effect.** Quark DSE:

$$S(p)^{-1} = Z_2\,(i\not{p} + m_0) + \Sigma(p)\,, \qquad \Sigma(p) = -\frac{4g^2}{3}\,Z_\Gamma \int \frac{d^4q}{(2\pi)^4}\, i\gamma^\mu\, S(q)\, D^{\mu\nu}(k)\, \Gamma^\nu(l,k)$$

Quark DSE



$$S(p)^{-1} = Z_2\,(i\not{p} + m_0) + \Sigma(p)\,, \qquad \Sigma(p) = -\frac{4g^2}{3}\,Z_\Gamma \int \frac{d^4q}{(2\pi)^4}\, i\gamma^\mu\, S(q)\, D^{\mu\nu}(k)\, \Gamma^\nu(l,k)$$

- An illustrative model: gluon propagator = δ distribution Munczek, Nemirovsky, Phys. Rev. D 28 (1983)

$$\frac{4}{3}\,\frac{g^2}{(2\pi)^4}\,D^{\mu\nu}(k) \to \Lambda^2\delta^4(k)\,\delta^{\mu\nu} \quad \Rightarrow \quad S(p)^{-1} = i\not{p} + m_0 + \Lambda^2\,\gamma^\mu\,S(p)\,\gamma^\mu \quad \Rightarrow \quad A(p^2) = 1 + \frac{2\Lambda^2}{A(p^2)\,(p^2 + M(p^2)^2)}\,,$$

$$A(p^2)M(p^2) = m_0 + 2M(p^2)\,\frac{2\Lambda^2}{A(p^2)\,(p^2 + M(p^2)^2)}$$

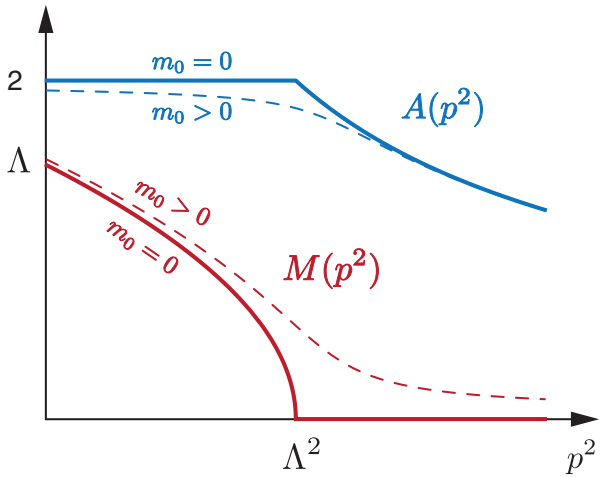
- Chiral limit ($m_0 = 0$):

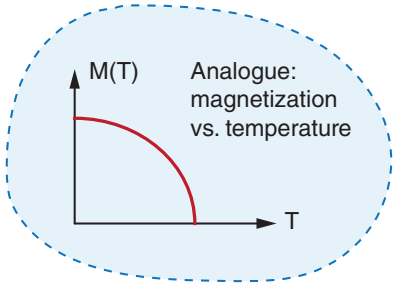
  Chirally symmetric solution:

$$A(p^2) = \frac{1}{2}\left(1 + \sqrt{1 + \frac{8\Lambda^2}{p^2}}\right), \qquad M(p^2) = 0$$

  Chiral-symmetry breaking solution:

$$A(p^2) = 2\,, \qquad M(p^2) = \sqrt{\Lambda^2 - p^2}$$

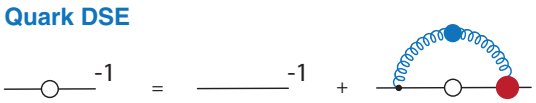




- In realistic DSE solution, no sharp phase transition in chiral limit, but otherwise similar



Quark DSE



$$S(p)^{-1} = Z_2\,(i\not{p} + m_0) + \Sigma(p)\,, \qquad \Sigma(p) = -\frac{4g^2}{3}\,Z_\Gamma \int \frac{d^4q}{(2\pi)^4}\; i\gamma^\mu\, S(q)\, D^{\mu\nu}(k)\,\Gamma^\nu(l,k)$$

- **Renormalization**:

$$\Sigma(p) = i\not{p}\,\Sigma_A(p^2) + \Sigma_M(p^2) \qquad \Rightarrow \qquad \begin{aligned} A(p^2) &= Z_2 + \Sigma_A(p^2)\,, \\ M(p^2)A(p^2) &= Z_2\, m_0 + \Sigma_M(p^2) \end{aligned}$$

Self-energy integrals are logarithmically UV-divergent
⇒ **regularize** by integrating up to cutoff,
⇒ **renormalize** by setting 2 boundary conditions: $A(\mu^2) \stackrel{!}{=} 1\,, \qquad M(\mu^2) \stackrel{!}{=} m$

$$Z_2 = 1 - \Sigma_A(\mu^2)\,, \qquad m_0 = \frac{m - \Sigma_M(\mu^2)}{1 - \Sigma_A(\mu^2)}\,,$$

$$A(p^2) = 1 + \Sigma_A(p^2) - \Sigma_A(\mu^2)\,,$$

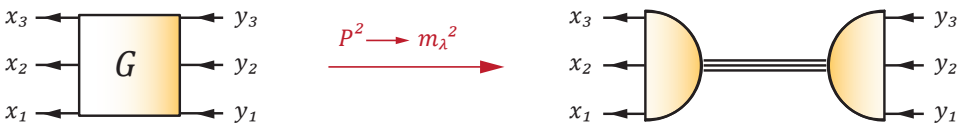
$$M(p^2)A(p^2) = m + \underbrace{\Sigma_M(p^2) - \Sigma_M(\mu^2)}_{\text{finite}}$$

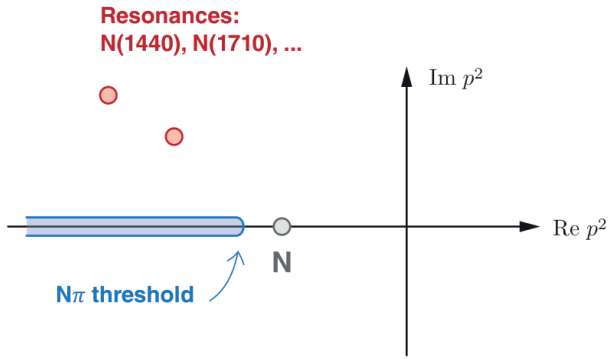
Divergences cancel in subtraction!
Renormalized $A(p^2)$ now depends on renormalization scale μ, but $M(p^2)$ does not

- DSE can be solved **iteratively**: Start with guess for $A(p^2)$, $M(p^2)$, e.g., set them to 1. Calculate self-energy integrals & evaluate them at renormalization scale. Determine Z_2. Calculate $A(p^2)$, $M(p^2)$ again. Proceed until converged.

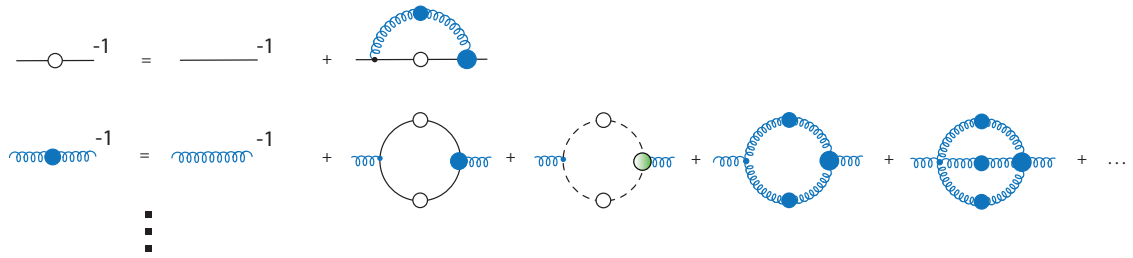
Hadrons?

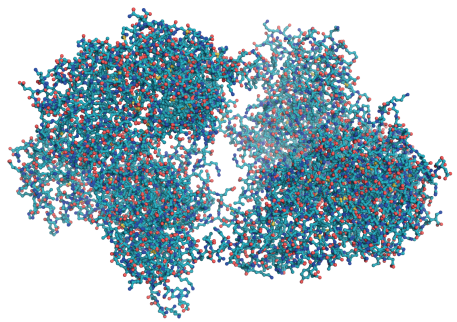
Baryon poles appear in elementary quark 6-point function:





How to calculate this? In principle from DSEs:

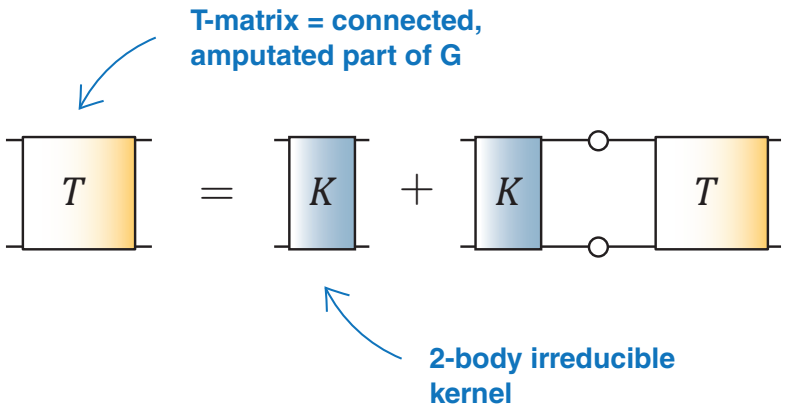




complicated structure & equations
for higher n-point functions,
more efficient: solve
Bethe-Salpeter equations

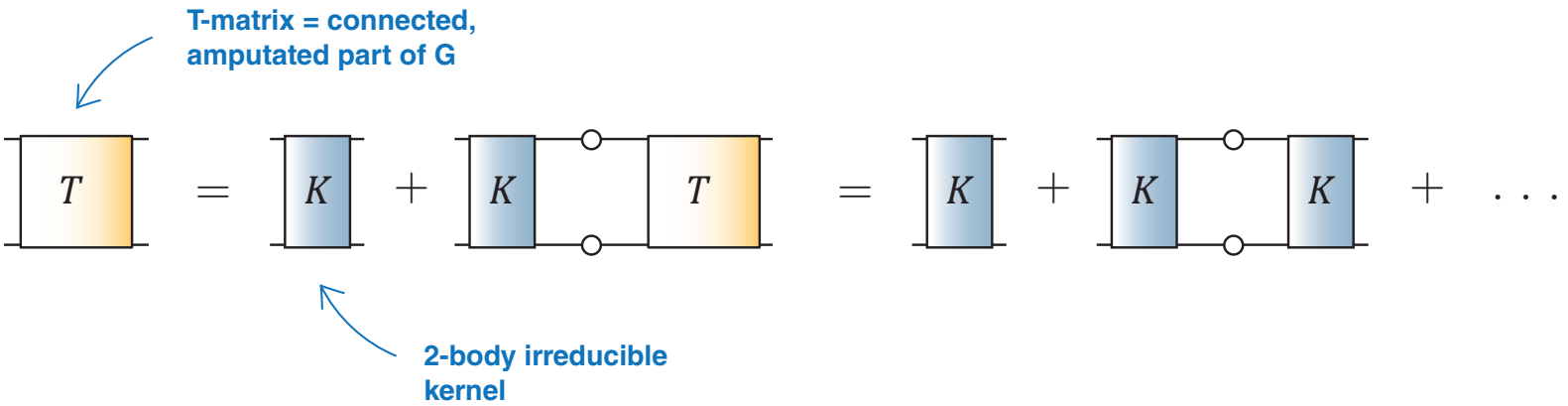
Bethe-Salpeter equations

Write down **inhomogeneous BSE**:



Bethe-Salpeter equations

Write down **inhomogeneous BSE**:



Analogy: **geometric series**

$$
\begin{aligned}
f(x) &= 1 + x\, f(x) \\
&= 1 + x + x^2\, f(x) \\
&= 1 + x + x^2 + x^3\, f(x)
\end{aligned}
\qquad \Rightarrow \quad f(x) = \frac{1}{1-x}
$$

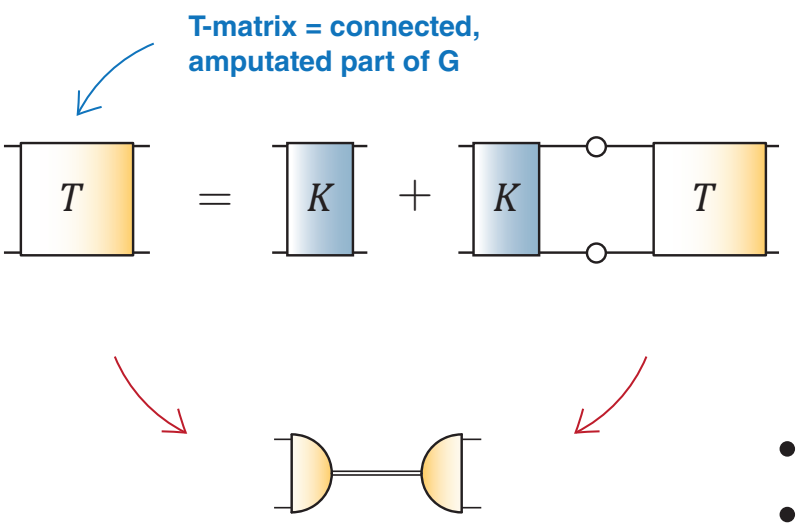
"non-perturbative"

$$
f(x) \approx 1 + x + x^2 + x^3 + \ldots \qquad \text{only for} \quad |x| < 1
$$

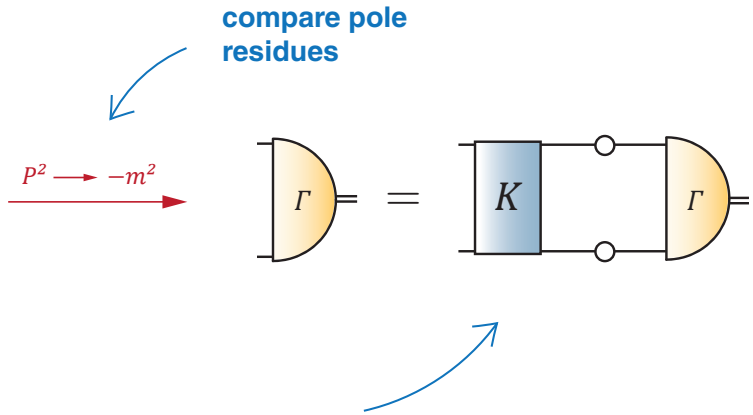
"perturbative"

Bethe-Salpeter equations

Write down **inhomogeneous BSE**:



Homogeneous BSE at pole:



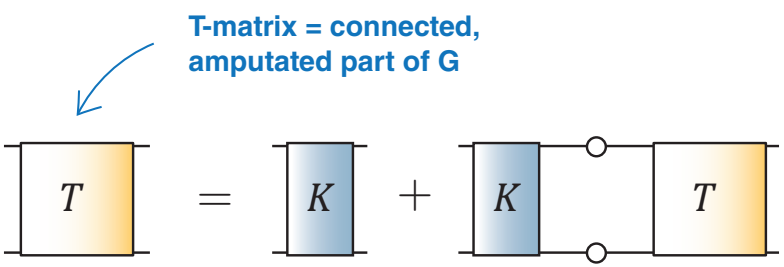
- $q\bar{q}$ irreducible kernel
- chiral symmetry constraints (V + AV WTI)
- can be systematically derived from effective action, depends on QCD's n-point functions



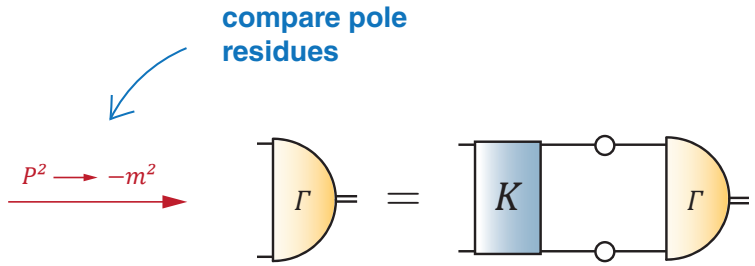
- Analogue of Schrödinger equation in QFT!
- Γ = **Bethe-Salpeter amplitude**

Bethe-Salpeter equations

Write down **inhomogeneous BSE**:

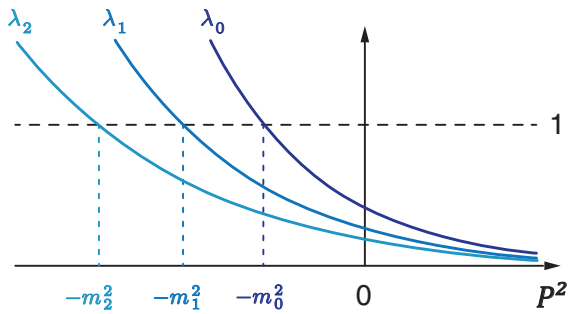


Homogeneous BSE at pole:



BSE = eigenvalue equation,
pole in T $\Leftrightarrow$ **eigenvalue = 1**

$$K G_0 \Gamma_i = \lambda_i \Gamma_i$$



Explicitly:

$$\Gamma(p,P) = \int \frac{d^4q}{(2\pi)^4} \mathbf{K}_{\alpha\gamma,\delta\beta}(p,q,P)\,[S(q_+)\,\Gamma(q,P)\,S(q_-)]_{\gamma\delta}$$

Basis decomposition:

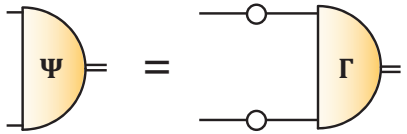
$$\Gamma(p,P) = \sum_{i=1}^{n} f_i(p^2, \hat{p}\cdot\hat{P}, P^2)\,\tau_i(p,P)$$

$\Rightarrow$ Coupled Lorentz-invariant equations
for the dressing functions f_i

Chiral symmetry and the pion

Maris, Roberts, Tandy, PLB 420 (1998)

- **Bethe-Salpeter WF** of pion: $[i\Psi(p,P)]_{\alpha\beta} = \int d^4x\, e^{-ipx} \langle 0 | \mathsf{T}\, \psi_\alpha(x)\, \bar\psi_\beta(0) | P, n\rangle$

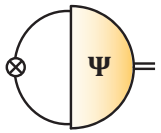


- **Axialvector and pseudoscalar currents:**

pion electroweak decay constant

$$j_5^\mu(x) = Z_2\, \bar\psi(x)\, i\gamma^\mu\gamma_5\, \psi(x)\,, \qquad \langle 0 | j_5^\mu(x) | P\rangle = \langle 0 | j_5^\mu(0) | P\rangle\, e^{ix\cdot P} = i f_\pi\, P^\mu e^{ix\cdot P}\,,$$
$$j_5(x) = Z_2 Z_m\, \bar\psi(x)\, i\gamma_5\, \psi(x)\,, \qquad \langle 0 | j_5(x) | P\rangle = \langle 0 | j_5(0) | P\rangle\, e^{ix\cdot P} = r_\pi\, e^{ix\cdot P}\,.$$

$$\Rightarrow \quad f_\pi\, P^\mu = Z_2 \int \frac{d^4p}{(2\pi)^4}\, \mathrm{Tr}\,\{i\gamma_5\gamma^\mu \Psi(p,P)\}\,, \qquad r_\pi = Z_2 Z_m \int \frac{d^4p}{(2\pi)^4}\, \mathrm{Tr}\,\{\gamma_5 \Psi(p,P)\}$$



- Now plug in **PCAC relation** from classical Lagrangian:

$$\partial_\mu j_5^\mu = 2m j_5 \quad \Rightarrow \quad f_\pi\, m_\pi^2 = 2m r_\pi \qquad \Rightarrow \text{ if } f_\pi \text{ does } \textbf{not} \text{ vanish in chiral limit, } m_\pi \text{ must be zero!}$$

- How can one see this? From **axialvector WTI** (= PCAC for 3-point functions) in chiral limit:

$$f_\pi\, \Gamma(p,0) = A(p^2)\, M(p^2)\, \gamma_5$$

We know that $M(p^2) \neq 0$ in chiral limit $\Rightarrow f_\pi \neq 0 \Rightarrow m_\pi = 0$. **Goldstone theorem!**

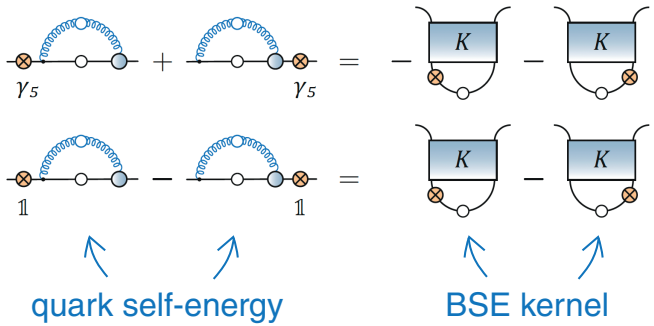
Take trace with $S(p)\,\gamma_5\, S(p)$ $\Rightarrow$ $f_\pi\, r_\pi = -\dfrac{\langle\bar\psi\psi\rangle}{N_f} = Z_2 Z_m\, N_c \displaystyle\int \frac{d^4p}{(2\pi)^4}\, \mathrm{Tr}\, S(p) = Z_2 Z_m \frac{N_c}{(2\pi)^2} \int dp^2\, \frac{p^2}{A(p^2)}\, \frac{M(p^2)}{p^2 + M(p^2)^2}$

Plug this back into PCAC relation $\Rightarrow$ $f_\pi^2 m_\pi^2 = -2m\,\langle\bar\psi\psi\rangle / N_f$ **Gell-Mann-Oakes-Renner relation**

Chiral symmetry and the pion

So far, everything exact ⇒ pion from BSE is automatically **massless in chiral limit** if

- **BSE kernel respects axialvector WTI**
- Chiral symmetry is dynamically broken such that $M(p^2) \neq 0$ in chiral limit (automatic in DSE)

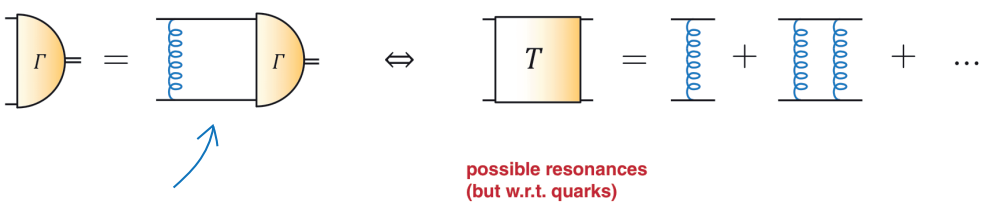


... axialvector WTI

... vector WTI

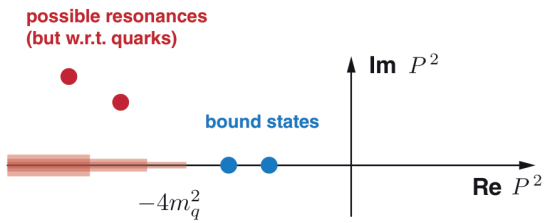
⇒ BSE kernel and quark propagator **must be related!**

- Let's try a gluon **ladder kernel:**



free propagators

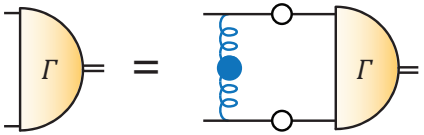
$$\frac{-i\not{p}+m}{p^2+m^2}$$



- breaks chiral symmetry
- generates bound-state poles in T, possibly also resonances
- but also quark thresholds & cuts: "hadrons" decay into quarks, no confinement
- would be ok if elementary d.o.f. were not quarks but **hadrons** (→ EFTs)

Chiral symmetry and the pion

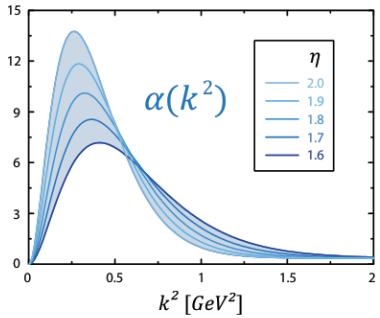
- Better: **rainbow-ladder**
  Maris, Roberts, PRC 56 (1997),
  Maris, Tandy, PRC 60 (1999),
  Qin, Chang et al., PRC 84 (2011)

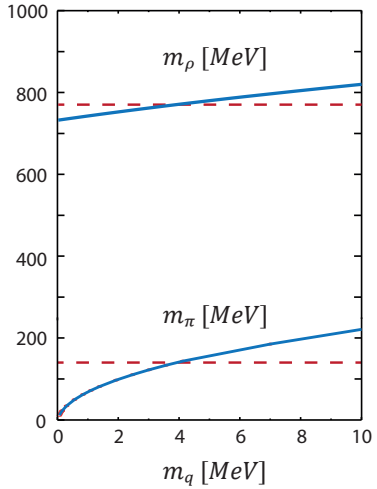


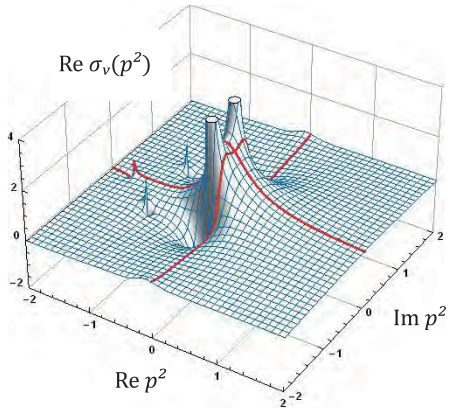
dressed propagators
from quark DSE

$$\frac{1}{A(p^2)}\,\frac{-i\not{p}+M(p^2)}{p^2+M(p^2)^2}$$

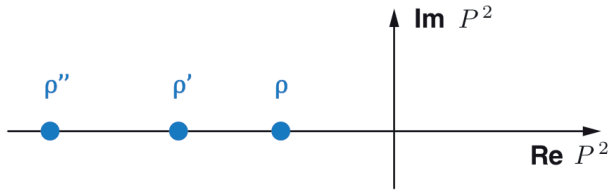
Kernel = effective gluon exchange
with effective interaction







- chiral symmetry automatic, **massless pion** in chiral limit ✓
- quark propagators do not develop real poles
  ⇒ no quark thresholds ✓



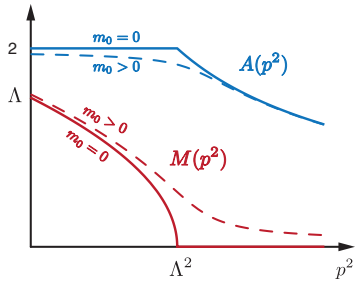
- but also no resonances, instead **bound states**
  (need to go beyond rainbow-ladder)

Chiral symmetry and the pion

Consider again our earlier model
(gluon propagator = δ distribution)

Munczek, Nemirovsky, Phys. Rev. D 28 (1983)

$$\frac{4}{3}\,\frac{g^2}{(2\pi)^4}\,D^{\mu\nu}(k) \to \Lambda^2\delta^4(k)\,\delta^{\mu\nu}$$



$$A(p^2) = 2\,,$$

$$M(p^2) = \sqrt{\Lambda^2 - p^2}$$

- Quark condensate:

$\Lambda = 1$ GeV

$$-\langle\bar{u}u\rangle = \frac{N_c}{(2\pi)^2}\int_0^{\Lambda^2} dp^2\,p^2\,\frac{\sqrt{\Lambda^2-p^2}}{2\Lambda^2} = \frac{2}{15}\,\frac{N_c}{(2\pi)^2}\,\Lambda^3 \approx (220\,\text{MeV})^3$$

not bad!

- Meson BSE for $P^2 = 0 \Rightarrow P^\mu = 0 \Rightarrow$ pion amplitude has only 1 tensor: $\Gamma(p) = f_1(p^2)\,\gamma_5$

$$\Gamma(p) = \frac{4g^2}{3}\int\frac{d^4q}{(2\pi)^4}\,i\gamma^\mu S(q)\,\Gamma(q)\,S(q)\,i\gamma^\nu\,D^{\mu\nu}(k) \;\to\; -\Lambda^2\,\gamma^\mu S(p)\,\Gamma(p)\,S(p)\,\gamma^\mu$$

$$\Rightarrow \quad 1 = \Lambda^2\,\gamma^\mu\gamma_5\,S(p)\,\gamma_5\,S(p)\gamma^\mu = \Lambda^2\gamma^\mu S(-p)\,S(p)\,\gamma^\mu = \frac{4\Lambda^2}{A(p^2)^2\,(p^2+M(p^2)^2)}$$

yes!

BSE has eigenvalue 1 at $P^2 = 0 \Rightarrow$ **massless pion in chiral limit**

- **Pion decay constant**: use $f_1(p^2) = A(p^2)M(p^2)/f_\pi$

$$f_\pi^2 = \frac{N_c\,Z_2}{4\pi^2}\int dp^2\,p^2\,\frac{M(p^2)}{A(p^2)}\,\frac{1}{(p^2+M(p^2)^2)^2}\left(M(p^2) - \frac{p^2}{4}\,\frac{dM(p^2)}{dp^2}\right)$$

... Pagels-Stokar formula, PRD 20 (1979)

Munczek-Nemirovsky model: $f_\pi = \sqrt{\frac{N_c}{2}}\,\frac{\Lambda}{4\pi} \approx 97$ MeV not bad!

$\Lambda = 1$ GeV

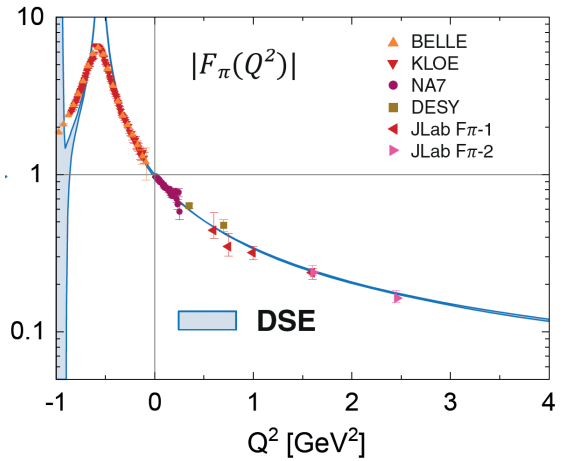
GMOR: $m_\pi^2 = \frac{32}{15}\,\Lambda m \approx 2\Lambda m$

Rainbow-ladder results

Rainbow-ladder has been very useful tool, many calculations over past > 25 years

Maris, Tandy, Nucl. Phys. B Proc. Suppl. 161 (2006),
Bashir, Chang, Cloet, El-Bennich, Liu, Roberts, Tandy, Comm. Th. Phys. 58 (2012),
GE, Sanchis-Alepuz, Williams, Alkofer, Fischer, Prog. Part. Nucl. Phys. 91 (2016),
. . . .

- **Pion electromagnetic form factor**

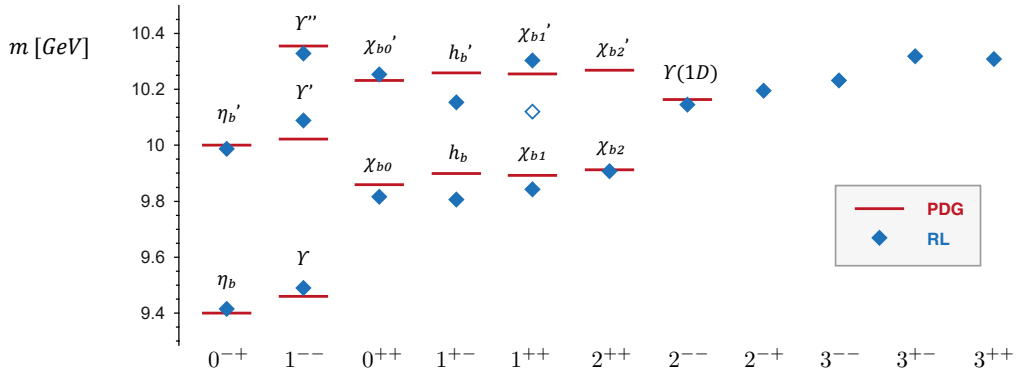


GE, Fischer, Williams, PRD 101 (2020)

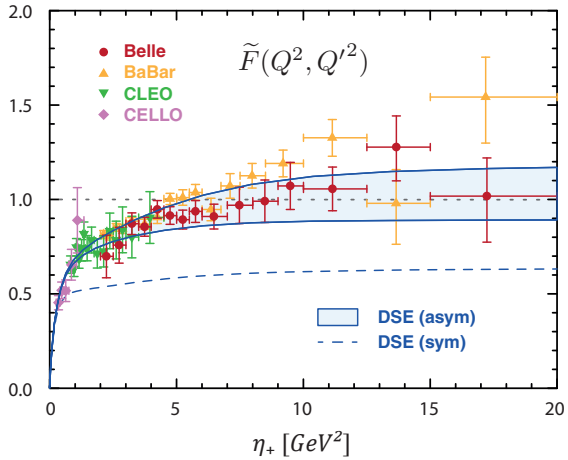
ρ pole: vector-meson dominance comes out automatically!

- **Bottomonium spectrum**

Fischer, Kubrak, Williams, EPJ A 51 (2015)



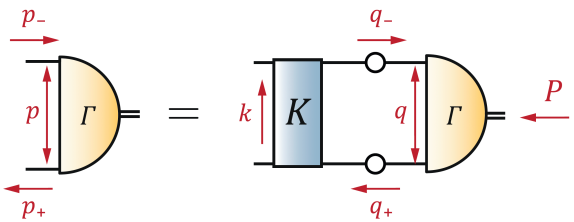
- **Pion → γγ transition form factor**



GE, Fischer, Weil, Williams, PLB 774 (2017)

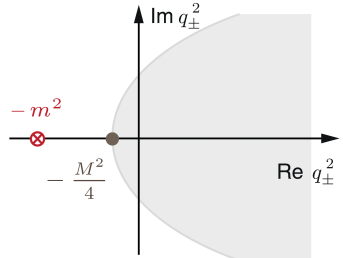
Quark in complex plane

For BSE, we usually need quark propagator for **complex momenta** ($p^2 \in \mathbb{C}$).



$P^2 = -M^2 \quad \Rightarrow$ **total momentum P^μ complex**

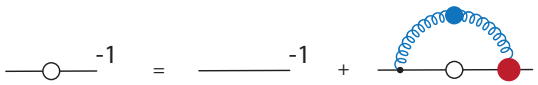
$q_\pm^\mu = q^\mu \pm \frac{P^\mu}{2} \quad \Rightarrow$ **quark momenta $q_\pm^\mu$ also complex**



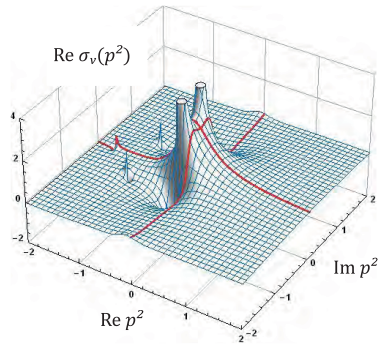
sampled on parabola in complex plane, limited by nearest quark singularities

How to do this?

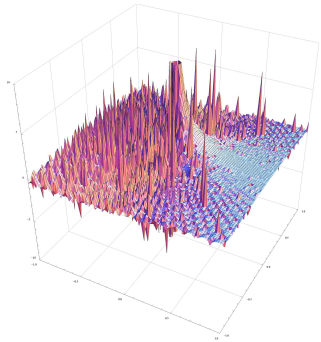
- For some interactions (e.g., Maris-Tandy) easy: put $p^2 \in \mathbb{C}$ in quark DSE and "iterate once more"



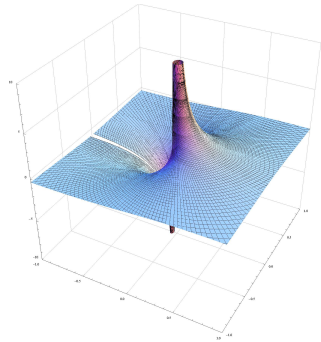
$$S(p)^{-1} = Z_2\,(i\not{p} + m_0) - \frac{4g^2}{3} Z_\Gamma \int \frac{d^4q}{(2\pi)^4}\, i\gamma^\mu\, S(q)\, D^{\mu\nu}(k)\, \Gamma^\nu(l,k)$$



- In general not possible: pole in gluon propagator produces cuts in integration. Naive integration over cuts yields nonsense
- Workaround: **analytic continuation** from real axis
- Solution: **contour deformations** or **Cauchy method**



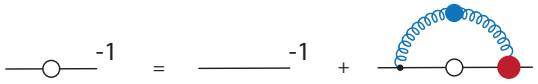
Naive calculation



Contour deformations

Quark in complex plane

Contour deformations:



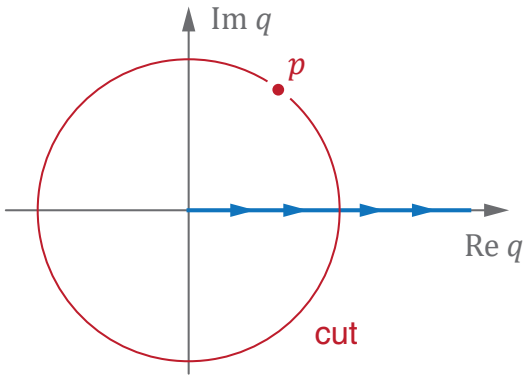
$$S(p)^{-1} = Z_2\,(i\not{p} + m_0) - \frac{4g^2}{3} Z_\Gamma \int \frac{d^4q}{(2\pi)^4}\, i\gamma^\mu\, S(q)\, D^{\mu\nu}(k)\, \Gamma^\nu(l,k)$$

- Integration: $\int d^4q\,(\dots) \;\to\; \int_0^\Lambda dq^2 \int_{-1}^{1} dz\,(\dots)$ $\Rightarrow$ q is real, $z \in [-1,1]$

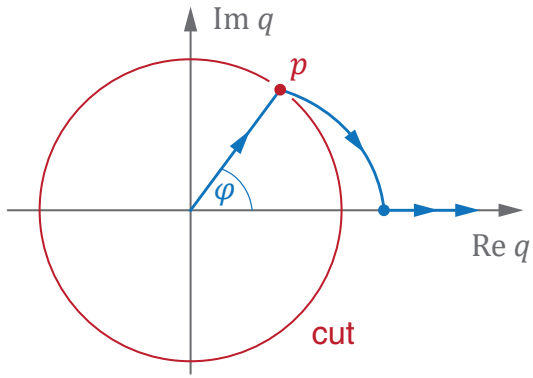
- Gluon momentum: $k^2 = (p-q)^2 = p^2 + q^2 - 2pqz \;\Rightarrow\; q^2 - 2pqz + p^2 - k^2 = 0 \;\Rightarrow\; q = p\left(z \pm i\sqrt{1 - z^2 - \frac{k^2}{p^2}}\right)$

  p is complex

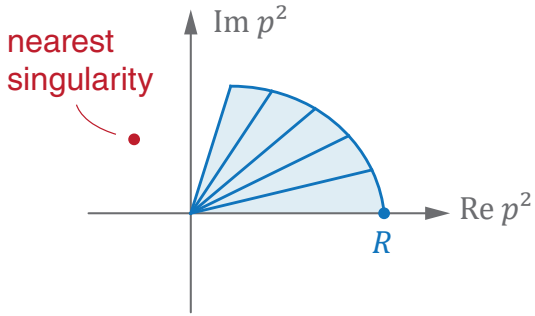
  $k^2 = 0$ $\Rightarrow$ circle in complex q plane with opening at $q = p$.



If integrand is singular at $k^2 = 0$: circular branch cut after z integration, cannot integrate q along real axis



Deform contour such that it passes through p. Because DSE is integral equation, we need quark propagator from previous iteration for these points in q $\Rightarrow$ solve DSE along this path! Ensures that you will never cross a cut



Can solve DSE in whole complex plane, up to angle where one hits the first dyn. generated singularity

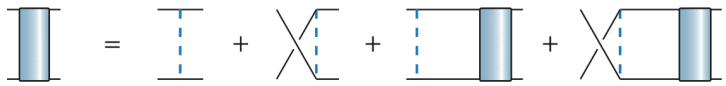
Quark in complex plane

Contour deformations:

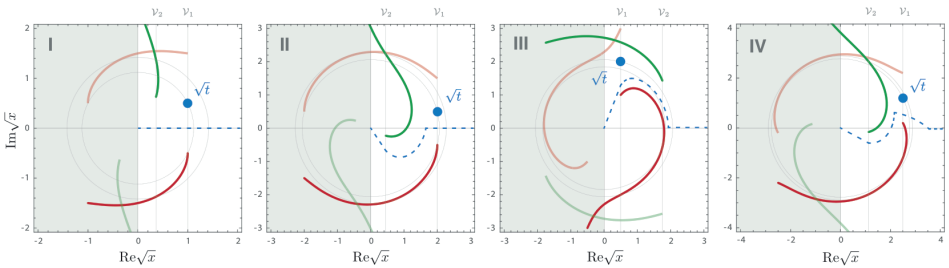
Fermion, gluon, ghost propagators, glueball correlator, rare pion decay, ρ and σ pole position, spectral functions, quark-photon vertex, timelike pion form factor, light-front wave functions, ...

Maris, PRD 52 (1995)
Alkofer, Detmold, Fischer, Maris, PRD 70 (2004)
GE, 0909.0703 [hep-ph]
Strauss, Fischer, Kellermann, PRL 109 (2012)
Windisch, Alkofer, Haase, Liebmann, CPC 184 (2013)
Windisch, Huber, Alkofer, PRD 87 (2013)
Weil, GE, Fischer, Williams, PRD 96 (2017)
Pawlowski, Strodthoff, Wink, PRD 98 (2018)
Williams, PLB 798 (2019)
Miramontes, Sanchis-Alepuz, EPJ A 55 (2019)
GE, Duarte, Pena, Stadler, PRD 100 (2019)
Fischer, Huber, PRD 102 (2020)
Santowsky, GE, Fischer, Wallbott, Williams, PRD 102 (2020)
Miramontes, Sanchis-Alepuz, Alkofer, PRD 103 (2021)
Santowsky, Fischer, EPJ C 82 (2022)
GE, Ferreira, Stadler, PRD 105 (2022)
...

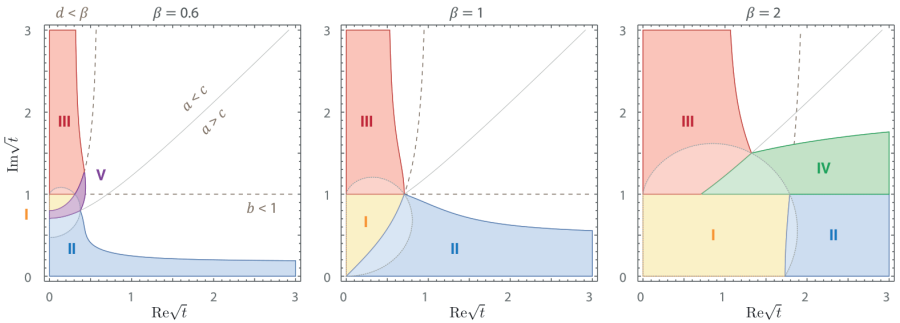
- **Scalar scattering equation:**
  direct access to 2nd sheet!



Cuts from K and G_0 in complex plane of integration variable:

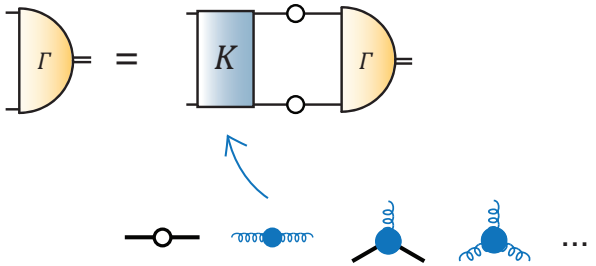


Can still cover **parts** of complex t plane:



Towards ab-initio

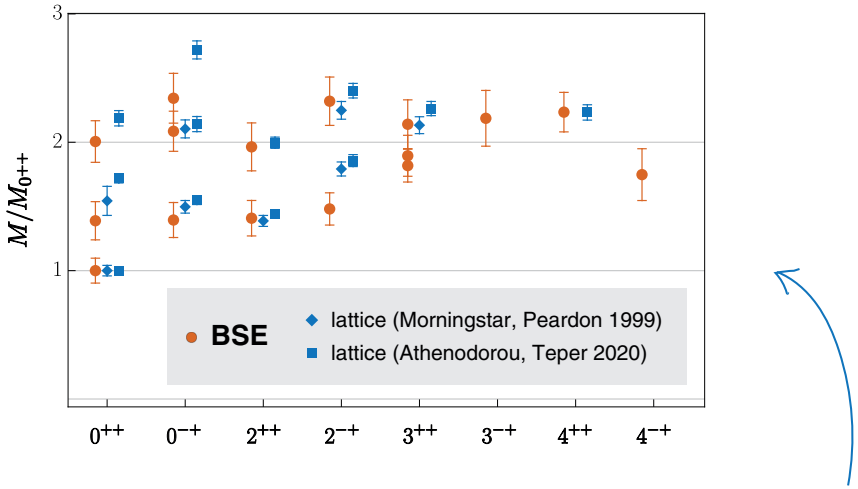
Go towards ab-initio calculations
by calculating **higher n-point functions**



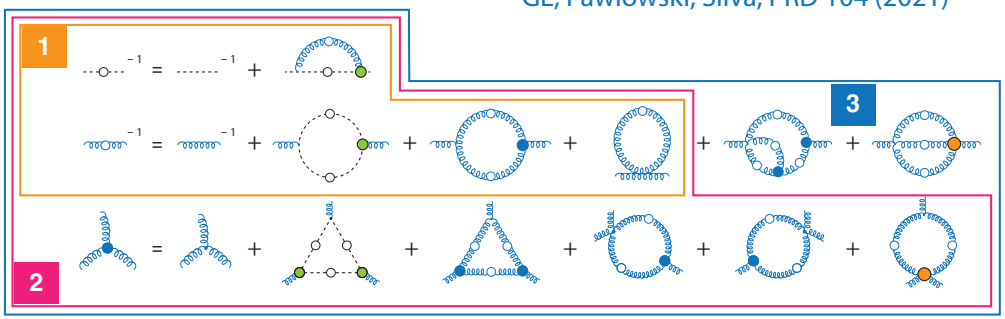
Various approaches on the market, e.g.:

- **Gauging quark DSE** yields vertex equation ⇒ meson poles
  Chang, Roberts, PRL 103 (2009)
- Expansion of BSE kernel via **nPI effective action**
  Williams, Fischer, Heupel, PRD 93 (2016)
- Including ππ decay channel generates **resonances**
  Santowsky et al., PRD 102 (2020)

- **Glueball spectrum** agrees with lattice QCD
  Huber, Fischer, Sanchis-Alepuz, EPJ C 80 (2020), EPJ C 81 (2021)



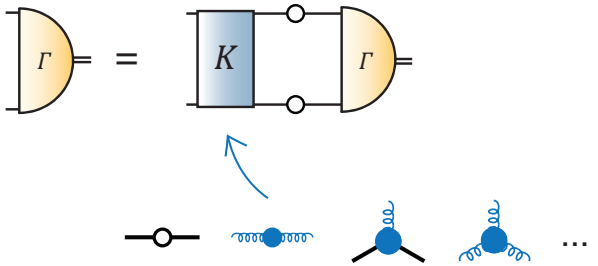
- Coupled **Yang-Mills DSEs** Huber, PRD 101 (2020), GE, Pawlowski, Silva, PRD 104 (2021)



Truncation error: 1 60% 2 10% 3 4%

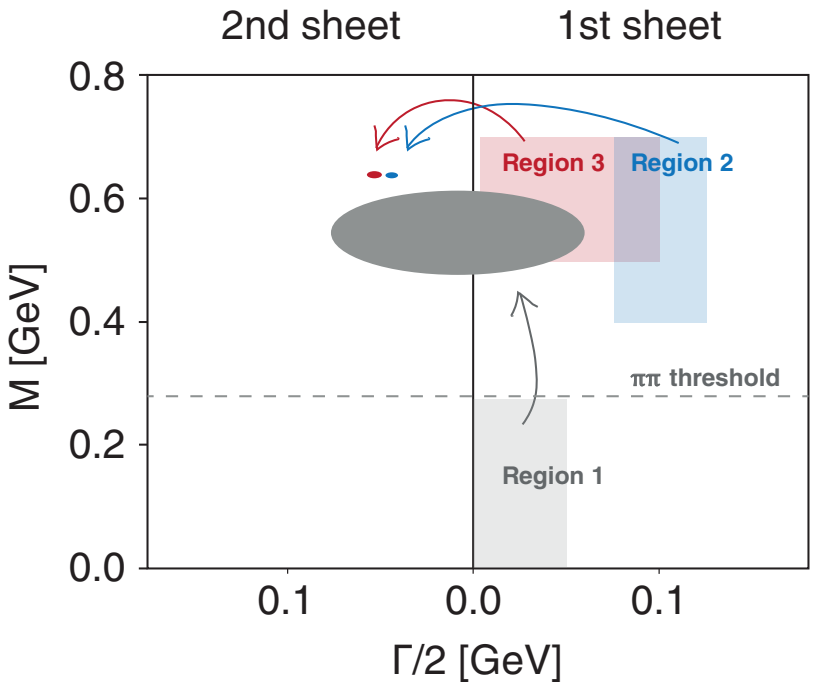
Towards ab-initio

Go towards ab-initio calculations
by calculating **higher n-point functions**



Various approaches on the market, e.g.:

- **Gauging quark DSE** yields vertex equation ⇒ meson poles
  Chang, Roberts, PRL 103 (2009)
- Expansion of BSE kernel via **nPI effective action**
  Williams, Fischer, Heupel, PRD 93 (2016)
- Including ππ decay channel generates **resonances**
  Santowsky et al., PRD 102 (2020)



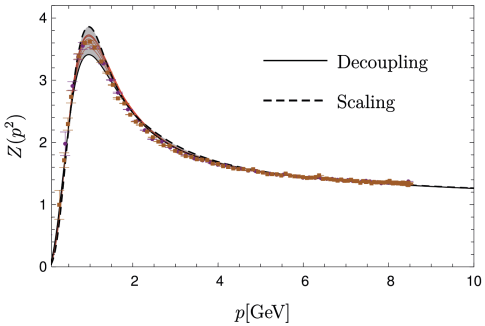
ρ resonance extracted from **homogeneous BSE:**

Solve ρ-meson BSE for complex total momenta on 1st sheet, analytically continue to 2nd sheet to determine pole resonance pole location

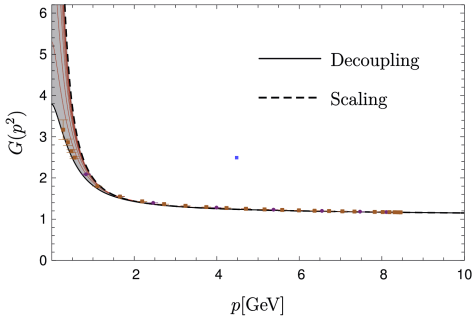
Towards precision

Markus Huber, ACHT 2025; Phys. Rept. 879 (2020) & PRD 101 (2020)

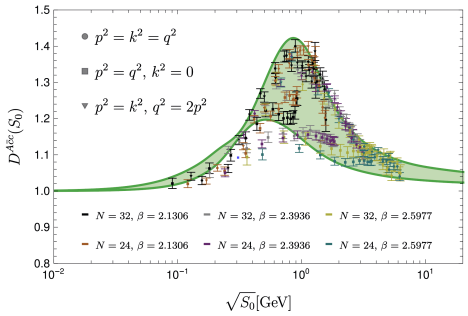
- Gluon propagator



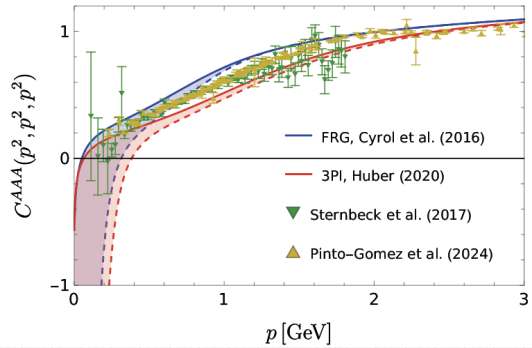
- Ghost propagator

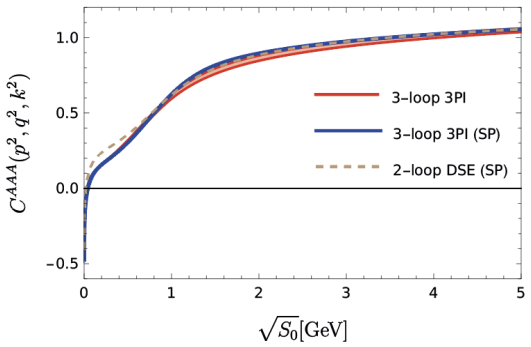


- Ghost-gluon vertex



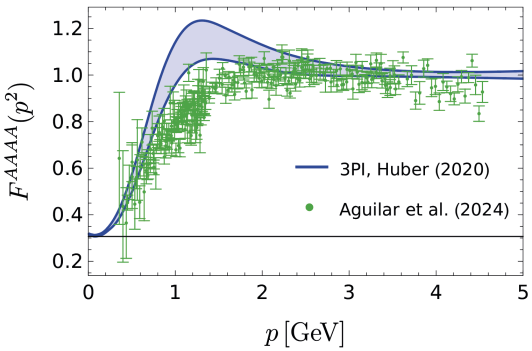
- Three-gluon vertex: lattice vs. FRG vs. DSE





Lattice: Pinto-Gómez, de Soto, Rodriguez-Quintero, PLB 838 (2023), Sternbeck et al., Pos LATTICE 2016; **FRG:** Cyrol et al., PRD 94 (2016); **DSE:** Huber, PRD 101 (2020), Huber, Fischer, Sanchis-Alepuz, 2503.03821

- Four-gluon vertex

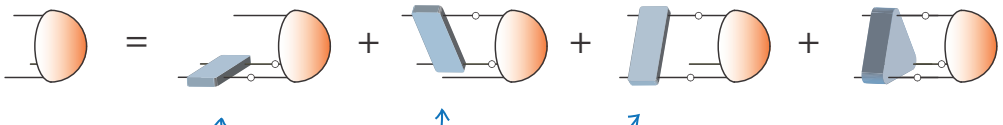


Lattice: Aguilar et al., PLB 858 (2024)
DSE: Huber, PRD 101 (2020)

Baryons

Three-quark BSE (Faddeev equation) for baryons:

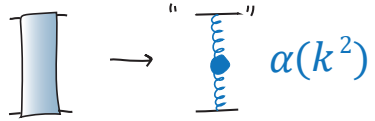
GE, Alkofer, Nicmorus, Krassnigg, PRL 104 (2010)

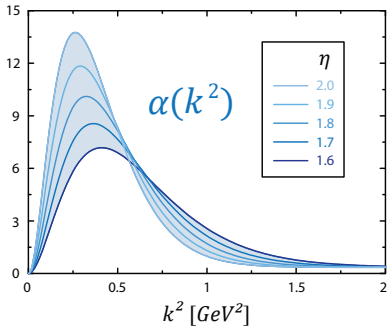


2-body kernel:
fixed in meson sector

3-body kernel:
Leading diagram (3-gluon vertex) vanishes by color trace, higher-order diagrams small (?)
2-quark correlations dominant?

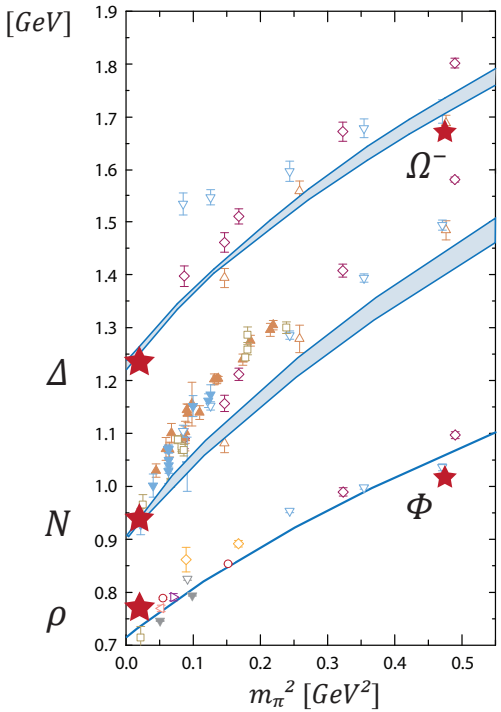
Rainbow-ladder





Scale set by f_π,
shape parameter → bands

Maris, Tandy, PRC 60 (1999)



- Analogous results for many **form factors**

  Review: GE, Sanchis-Alepuz, Williams, Alkofer, Fischer, Prog. Part. Nucl. Phys. 91 (2016)

- Relativistically, nucleon also has **p waves!**

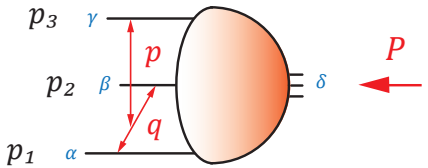


see also:
Qin, Roberts, Schmidt, PRD 97 (2018)

Baryons

Baryon's Faddeev amplitude:

GE, Alkofer, Nicmorus, Krassnigg, PRL 104 (2010)



$$\Psi_{\alpha\beta\gamma\delta}(p,q,P) = \sum_i f_i(p^2, q^2, p\cdot q, p\cdot P, q\cdot P)\ \tau_i(p,q,P)_{\alpha\beta\gamma\delta} \otimes \text{Flavor} \otimes \text{Color}$$

Lorentz-invariant **dressing functions**

Dirac-Lorentz tensors: 64 for J = 1/2, 128 for J = 3/2

group theory (same as before)

s	l	T_{ij}	
1/2	0	$\mathbb{1}\otimes\mathbb{1}$	s waves (8)
1/2	0	$\gamma_T^\mu\otimes\gamma_T^\mu$	
1/2	1	$\mathbb{1}\otimes\frac{1}{2}[\not p,\not q]$	p waves (36)
1/2	1	$\mathbb{1}\otimes\not p$	
1/2	1	$\mathbb{1}\otimes\not q$	
1/2	1	$\gamma_T^\mu\otimes\gamma_T^\mu\frac{1}{2}[\not p,\not q]$	
1/2	1	$\gamma_T^\mu\otimes\gamma_T^\mu\not p$	
1/2	1	$\gamma_T^\mu\otimes\gamma_T^\mu\not q$	
3/2	1	$3(\not p\otimes\not q-\not q\otimes\not p)-\gamma_T^\mu\otimes\gamma_T^\mu[\not p,\not q]$	
3/2	1	$3\not p\otimes\mathbb{1}-\gamma_T^\mu\otimes\gamma_T^\mu\not p$	
3/2	1	$3\not q\otimes\mathbb{1}-\gamma_T^\mu\otimes\gamma_T^\mu\not q$	
3/2	2	$3\not p\otimes\not p-\gamma_T^\mu\otimes\gamma_T^\mu$	d waves (20)
3/2	2	$\not p\otimes\not p+2\not q\otimes\not q-\gamma_T^\mu\otimes\gamma_T^\mu$	
3/2	2	$\not p\otimes\not q+\not q\otimes\not p$	
3/2	2	$\not q\otimes[\not q,\not p]-\frac{1}{2}\gamma_T^\mu\otimes[\gamma_T^\mu,\not p]$	
3/2	2	$\not p\otimes[\not p,\not q]-\frac{1}{2}\gamma_T^\mu\otimes[\gamma_T^\mu,\not q]$	

$$T_{ij}\,(\Lambda_\pm\gamma_5 C\otimes\Lambda_+)$$
$$(\gamma_5\otimes\gamma_5)\,T_{ij}\,(\Lambda_\pm\gamma_5 C\otimes\Lambda_+)$$

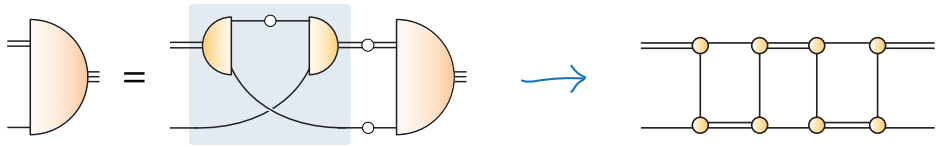
Positive-energy projector: $\Lambda_+ u = u$

- can be arranged into eigenstates of **S** and **L** in baryon's rest frame: s, p, d, f, ... waves
- **nonrel. quark model:** only 2 survive ($l=0$) $\left[\uparrow\downarrow\uparrow-\downarrow\uparrow\uparrow\right]$, $\left[\frac{1}{\sqrt{3}}(\uparrow\downarrow\uparrow+\downarrow\uparrow\uparrow-2\uparrow\uparrow\downarrow)\right]$, $\left[\uparrow\downarrow\downarrow-\downarrow\uparrow\downarrow\right]$, $\left[\frac{1}{\sqrt{3}}(\uparrow\downarrow\downarrow+\downarrow\uparrow\downarrow-2\downarrow\downarrow\uparrow)\right]$
- Relativistically, nucleon also has **p waves!**



Diquark correlations

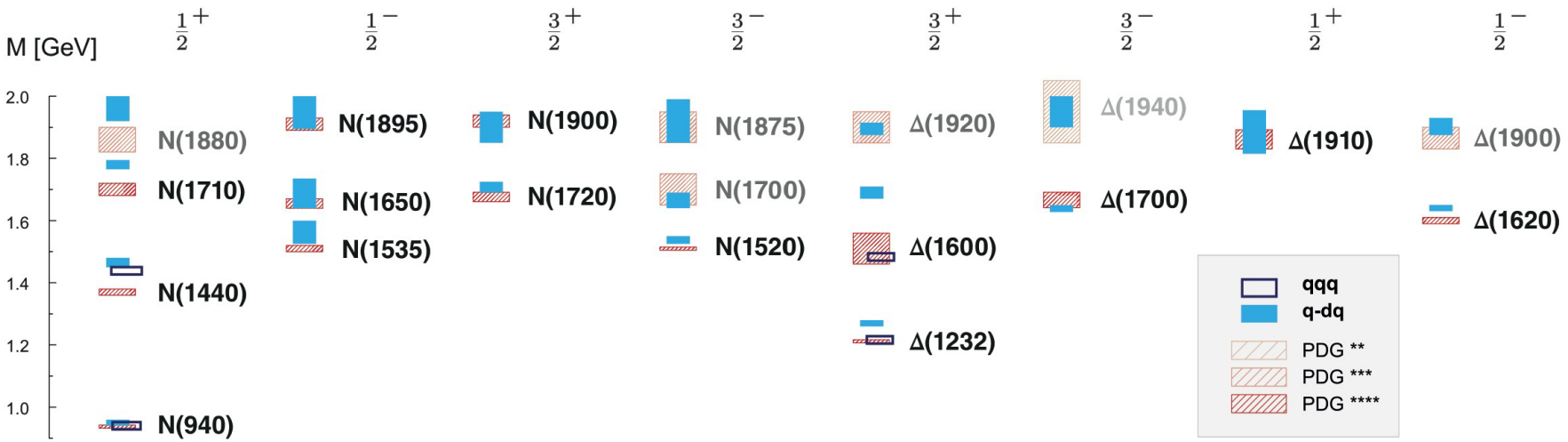
- **Quark-diquark** (two-body) equation
  Oettel et al., PRC 58 (1998), GE et al., Ann. Phys. 323 (2008), Cloet et al., FBS 46 (2009), Segovia et al., PRL 115 (2015)



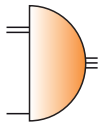
Diquark clustering in baryons?
Barabanov et al., Prog. Part. Nucl. Phys. 116 (2021)



- **Three-quark** and **quark-diquark** results very similar
  GE, Fischer, Sanchis-Alepuz, PRD 94 (2016)



Diquarks 101



Wave function: Dynamics $\otimes$ Flavor $\otimes$ Color

Flavor: $\mathcal{A}, \mathcal{S}$ — Color: $\mathcal{A}$

attractive

$$\mathbf{3} \otimes \mathbf{3} = \overline{\mathbf{3}} \oplus \mathbf{6}$$

Flavor:

$$\mathbf{2} \otimes \mathbf{2} = \mathbf{1}_A \oplus \mathbf{3}_S$$
$$\mathbf{3} \otimes \mathbf{3} = \overline{\mathbf{3}}_A \oplus \mathbf{6}_S$$
$$\mathbf{4} \otimes \mathbf{4} = \mathbf{6}_A \oplus \mathbf{10}_S$$

"good": $[ud]$; $[us]$, $[ds]$; $[uc]$, $[dc]$, $[sc]$

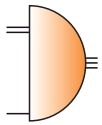
"bad": $\{uu\}$, $\{ud\}$, $\{dd\}$; $\{us\}$, $\{ds\}$, $\{ss\}$; $\{uc\}$, $\{dc\}$, $\{sc\}$, $\{cc\}$

I = 0
I = 1
I = 1/2

Quark content	q-dq	Isospin	contributes to
sss	s {ss}	0	Ω
ccc	c {cc}	0	Ω_c
uud	u [ud]	1/2	p
	u {ud}	1/2, 3/2	p, Δ^+
	d {uu}	1/2, 3/2	p, Δ^+
nns	n [ns]	0, 1	Λ, Σ
↑ n = u, d	n {ns}	0, 1	Λ, Σ
	s [nn]	0	Λ
	s {nn}	1	Σ

- Nucleon has "good" and "bad" diquarks, Delta only "bad"
- Λ: n [ns], n {ns}, s [nn]
  Σ: n [ns], n {ns}, s {nn}

Diquarks 101



Wave function: $\underbrace{\text{Dynamics}}_{\mathcal{A},\mathcal{S}} \otimes \underbrace{\text{Flavor}}_{\mathcal{A},\mathcal{S}} \otimes \underbrace{\text{Color}}_{\mathcal{A}} \stackrel{!}{=} \mathcal{A}$

- $\gamma_5 C, \ldots$ **scalar** $\quad \mathcal{A}: \ (\gamma_5 C)^T = -\gamma_5 C \quad \Leftrightarrow$ **"good"**
- $\gamma^\mu C, \ldots$ **axialvector** $\quad \mathcal{S}: \ (\gamma^\mu C)^T = \gamma^\mu C \quad \Leftrightarrow$ **"bad"**
- $C, \ldots$ **pseudoscalar** $\mathcal{A}$
- $\gamma^\mu \gamma_5 C, \ldots$ **vector** $\mathcal{A}$

} **"ugly"**

- ...

Total wave function:

$$\Gamma^{(\mu)}(q,P) = \sum_i \underbrace{f(q^2, q\cdot P, P^2 = -m^2)}_{\text{Dressing functions calculated from BSE}} \underbrace{\tau_i^{(\mu)}(q,P)}_{\text{Lorentz/Dirac tensors}} \otimes \text{Flavor} \otimes \text{Color}$$

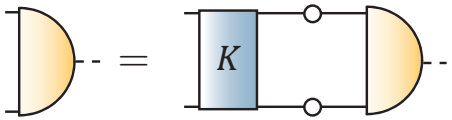
Diquark correlations

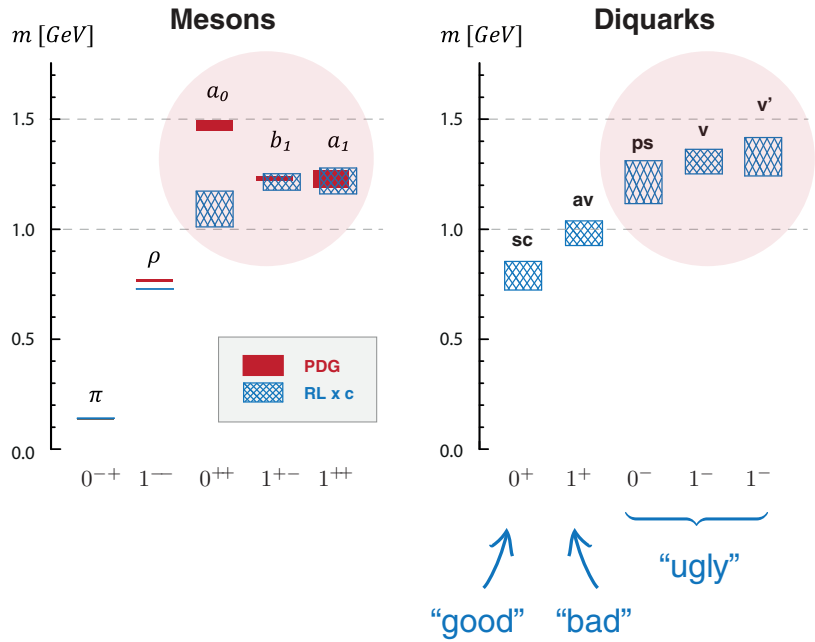
Mesons and diquarks are closely related:

attractive
$$3 \otimes \overline{3} = \mathbf{1} \otimes 8$$

attractive
$$3 \otimes 3 = \overline{3} \oplus 6$$

In BSE this comes out naturally: Maris, FBS 32 (2002)





Lowest-lying diquarks are dominant for ground-state octet & decuplet baryons

pseudoscalar mesons	⇔	**scalar diquarks** (~0.8 GeV)
vector mesons	⇔	**axialvector diquarks** (~1 GeV)

Higher-lying diquarks are subleading, but contribute to excited states & remaining channels

scalar mesons	⇔	**pseudoscalar diquarks** (~1.2 GeV)
axialvector mesons	⇔	**vector diquarks** (~1.3 GeV)

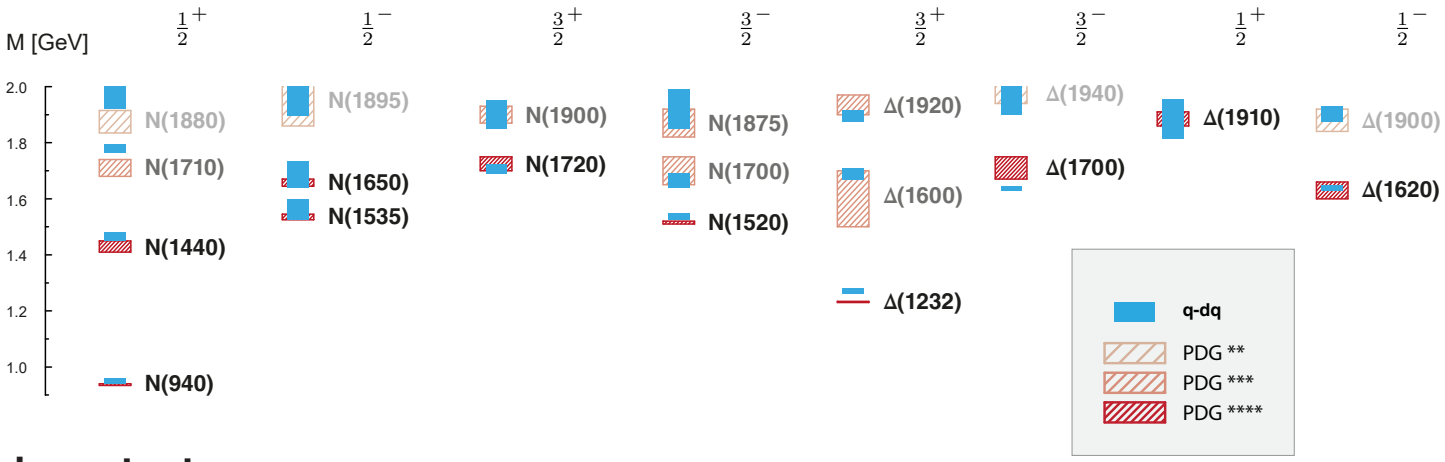
In RL, these are too strongly bound; simulate beyond-RL effects by (one) strength parameter c

Roberts, Chang, Cloet, Roberts, FBS 51 (2011)
GE, Fischer, Sanchis-Alepuz, PRD 94 (2016)

Diquark correlations

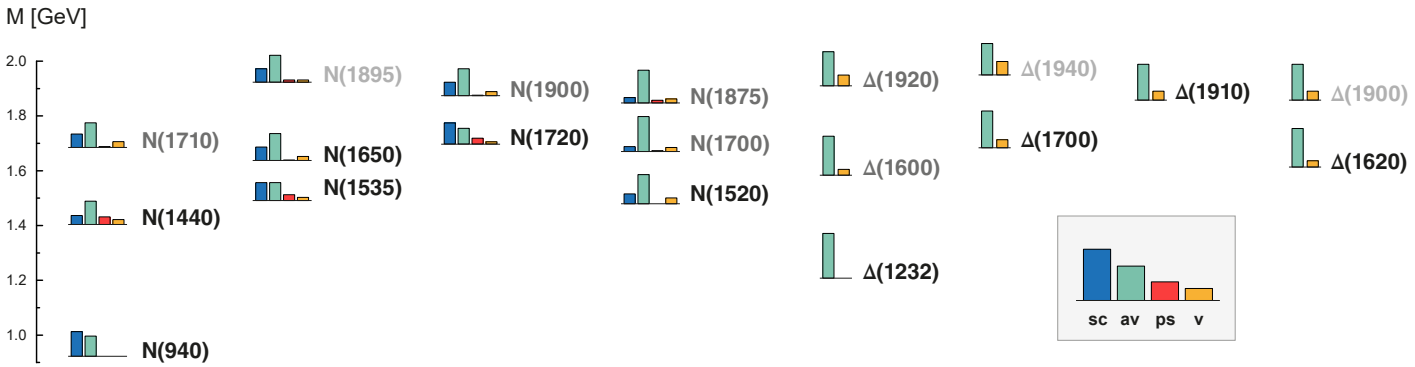
Light baryon spectrum

GE, Fischer, Sanchis-Alepuz, PRD 94 (2016)



Diquark content:

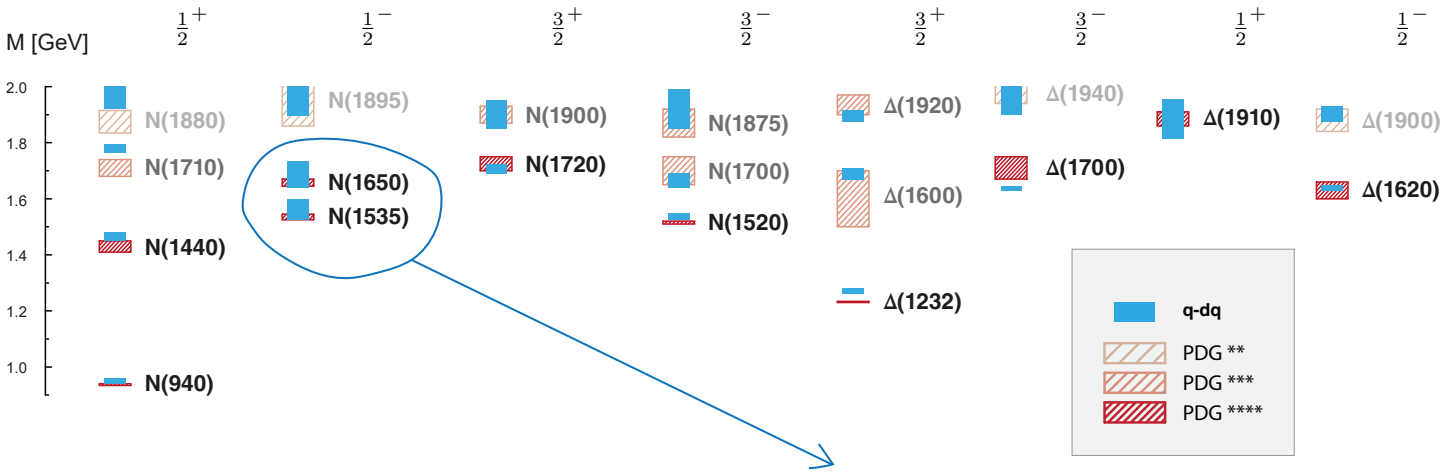
Barabanov et al., PPNP 116 (2021)



Diquark correlations

Light baryon spectrum

GE, Fischer, Sanchis-Alepuz, PRD 94 (2016)



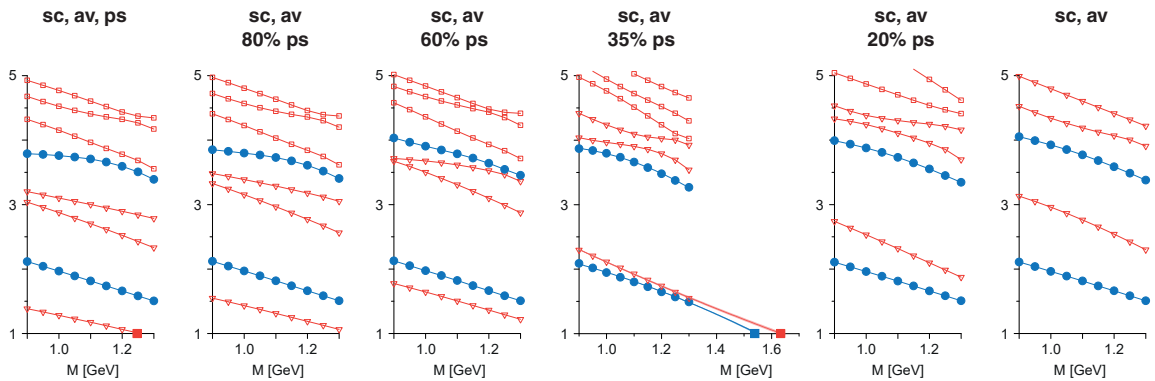
RL, all diquarks:
"N(1535)" too low

"Beyond RL":
N(1535), N(1650)

RL, sc+av only:
"N(1650)" too high

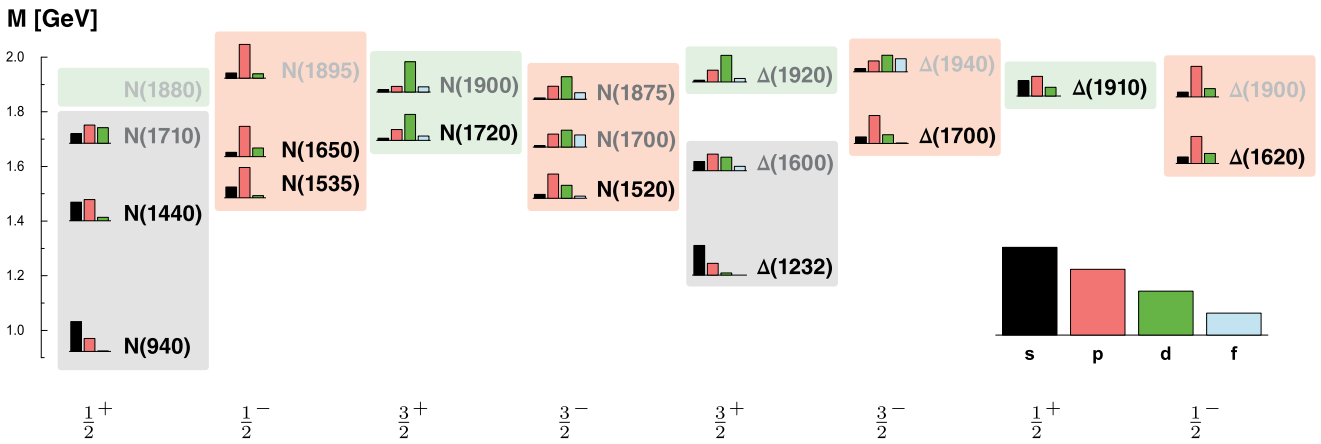
- **Level ordering** determined by diquark dynamics
- Diquarks are not pointlike, also here **rich spectrum!**

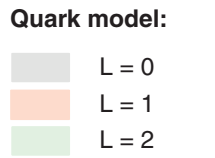
Barabanov et al., PPNP 116 (2021)



Relativistic effects

Orbital angular momentum: clear traces of nonrelativistic quark model, but strong relativistic effects (in some cases even dominant)

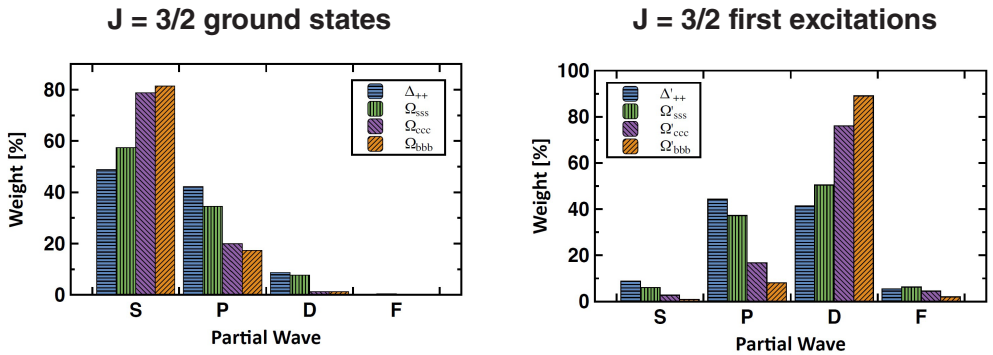




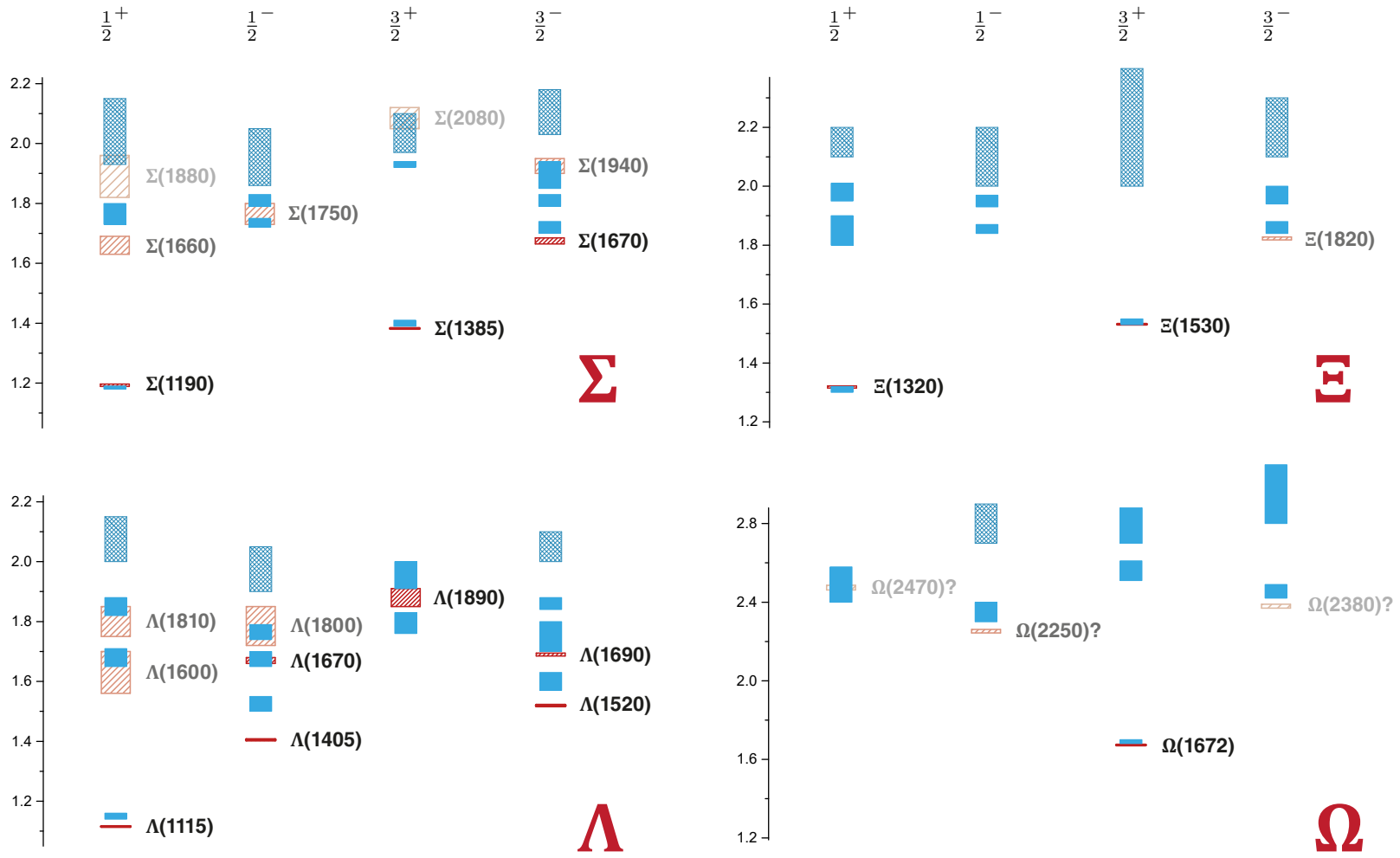
GE, FBS 58 (2017)

Relativistic contributions
even up to bottom baryons!

Qin, Roberts, Schmidt, PRD 97 (2018)

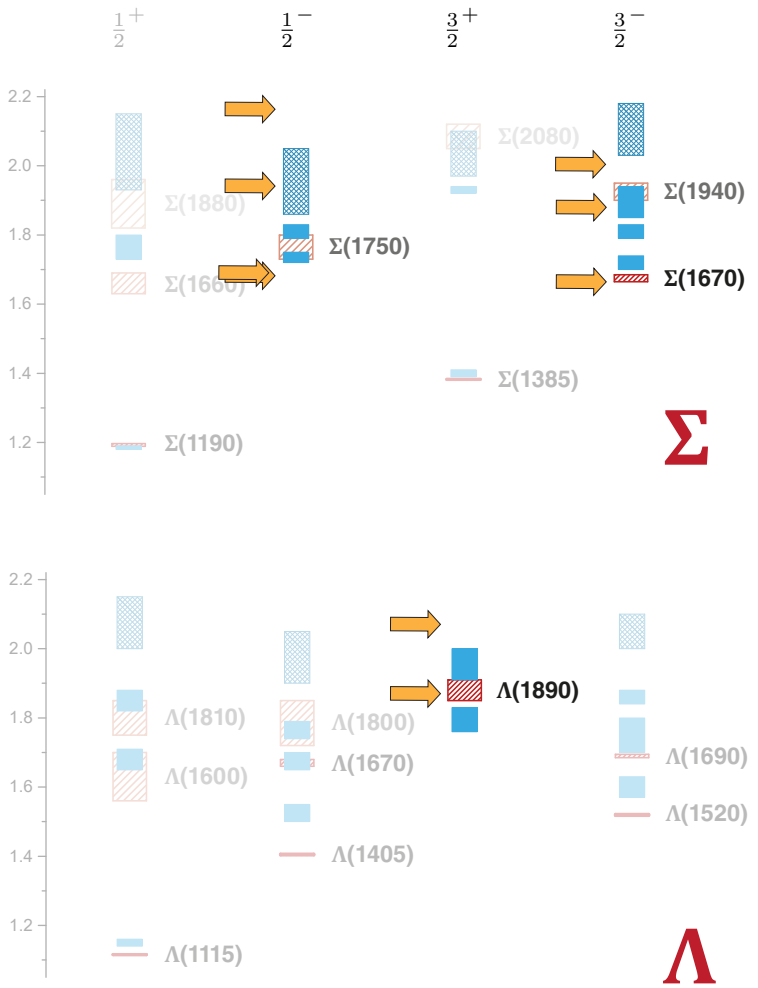


Strange baryons



GE, Fischer, FBS 60 (2019), Fischer, GE, PoS Hadron 2017

Strange baryons



New states from Bonn-Gatchina

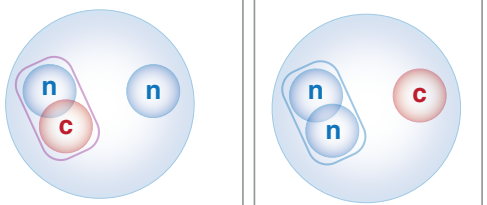
Sarantsev, Matveev, Nikonov, Anisovich, Thoma, EPJA 55 (2019)

GE, Fischer, FBS 60 (2019), Fischer, GE, PoS Hadron 2017

Heavy baryons

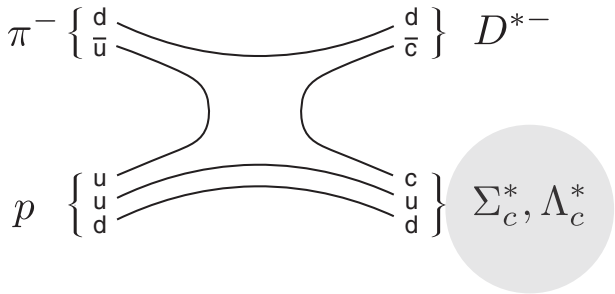
Quark content	q-dq	Isospin	contributes to
nnc	n [nc]	0, 1	Λ_c , Σ_c
↑ n = u, d	n {nc}	0, 1	Λ_c , Σ_c
	c [nn]	0	Λ_c
	c {nn}	1	Σ_c

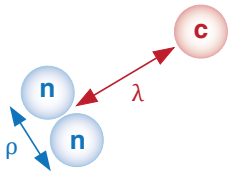
⇒ Λ_c: n [nc], n {nc}, c [nn]
Σ_c: n [nc], n {nc}, c {nn}



Sometimes these are assumed as dominant components, e.g. **J-PARC** charm baryon spectroscopy program (high-p)

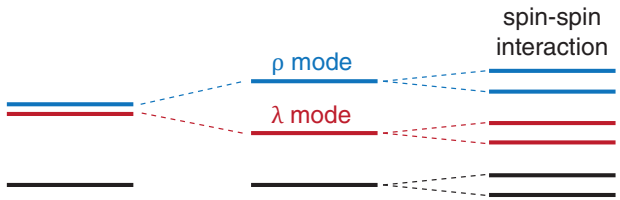
Kim, Hosaka, Kim, Noumi, Shirotori, PTEP 2014 (2014), 10, 103D01,
Shim, Hosaka, Kim, PTEP 2020 (2020) 5, 053D01





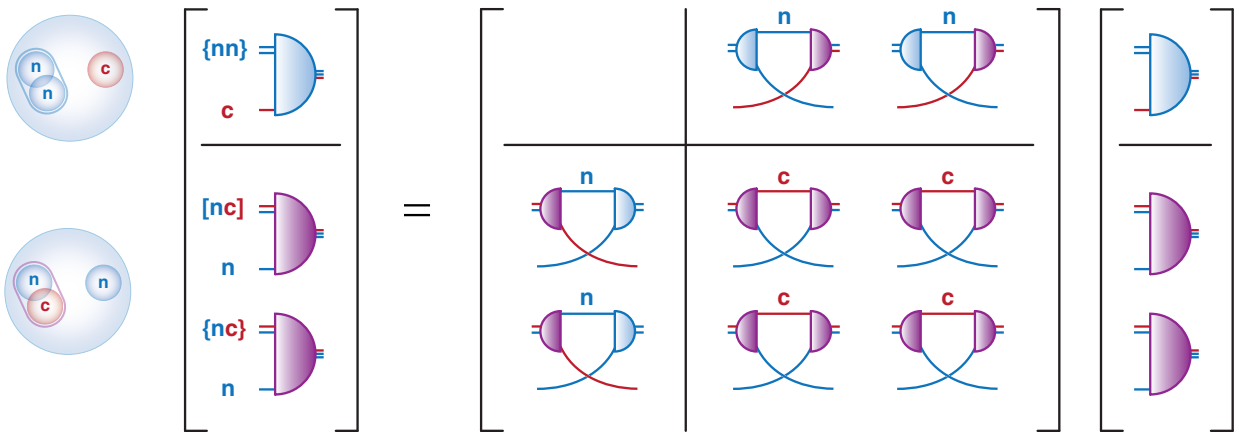
Decay of ρ mode → $\Sigma_c\pi$
Decay of λ mode → pD

Assumptions on spectrum & production rates



Heavy baryons

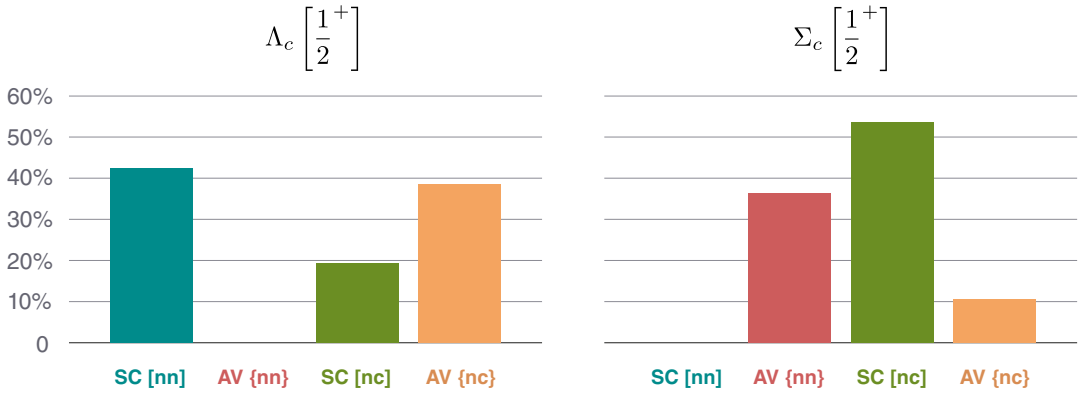
Quark-diquark BSE would not work under this assumption, e.g., Σ_c:

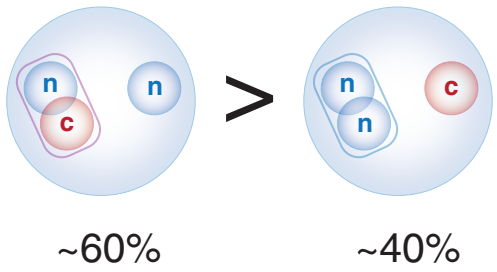


n[nc], n{nc} necessary, otherwise no equation. Presumably these are also dominant: cannot switch off n[nc], n{nc}, but c{nn}

Analogous for Λ_c and hyperons

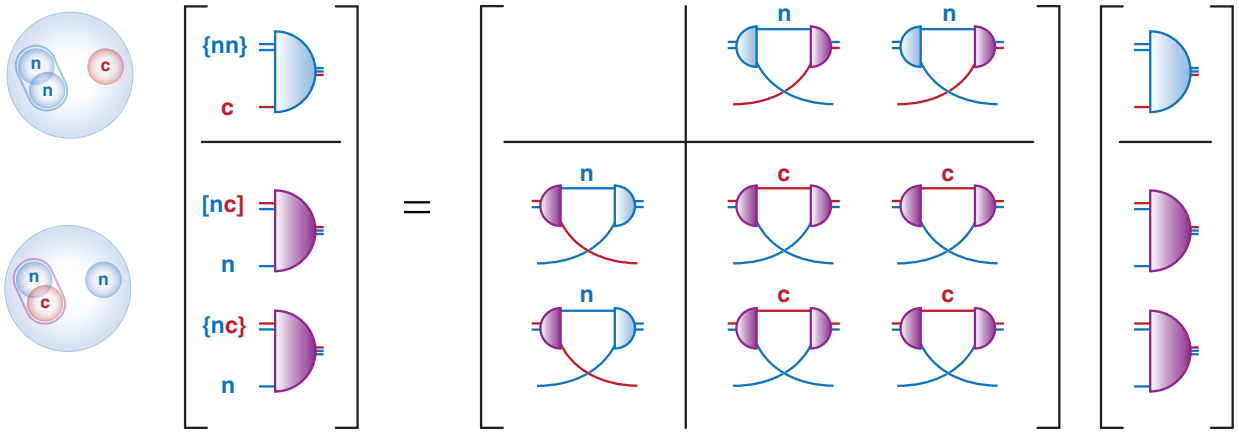
Results: **wave function contributions** Torcato, Arriaga, GE, Peña, FBS 64 (2023)





Heavy baryons

Quark-diquark BSE would not work under this assumption, e.g., Σ_c:

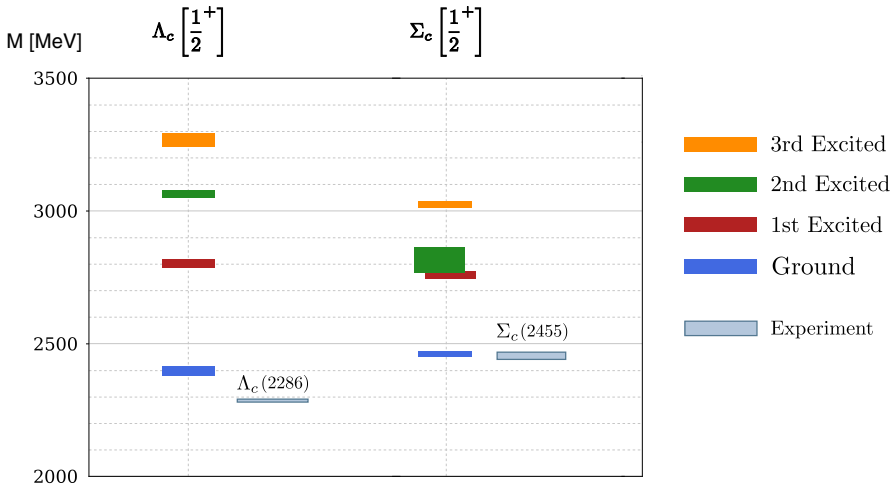


n[nc], n{nc} necessary, otherwise no equation. Presumably these are also dominant: cannot switch off n[nc], n{nc}, but c{nn}

Analogous for Λ_c and hyperons

Results: **spectrum**

Torcato, Arriaga, GE, Peña, FBS 64 (2023)

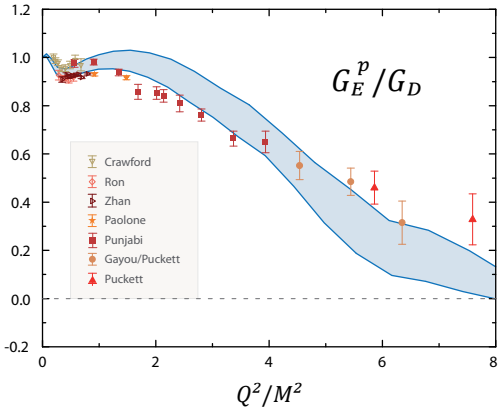


see also: Yin, Chen, Krein, Roberts, Segovia, PRD 100 (2019)

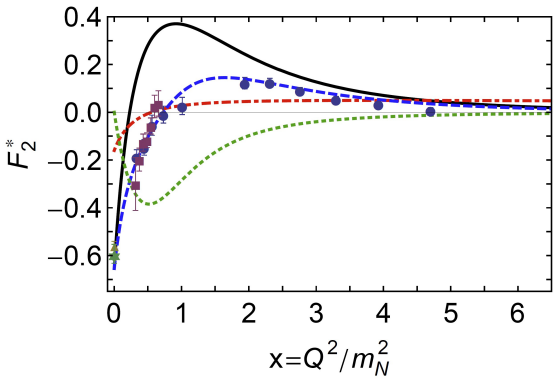
Form factors

Many **form factor calculations** in qqq or q(qq) approaches available:

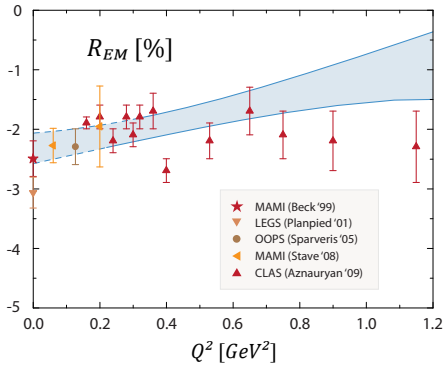
Nucleon electromagnetic FFs
GE, PRD 84 (2011)



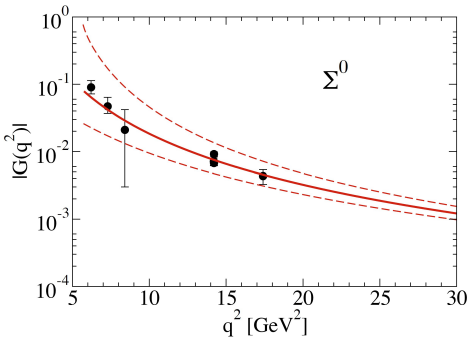
Roper em. transition FFs
Segovia et al., PRL 115 (2015)



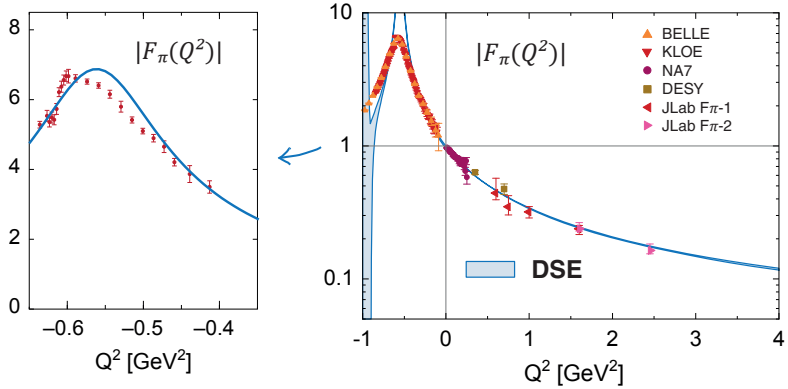
Δ-baryon em. transition FFs
GE, Nicmorus, PRD 85 (2012)



Timelike em. strangeness FFs
Ramalho, Peña, PRD 101 (2020)



Timelike pion FF dynamically develops ρ pole with $\pi\pi$ decay channel
Miramontes, Sanchis-Alepuz, Alkofer, PRD 103 (2021)



GE, Fischer, Williams, PRD 101 (2020),

1. Introduction
2. Spectrum & symmetries
3. Quark models
4. QFT toolbox
5. Functional methods
6. n-point functions & tensor bases
7. Multiquark states
8. Light-front wave functions

Lecture notes "QCD and hadron physics",
https://particle.uni-graz.at/en/quarks-hadrons-and-nuclei

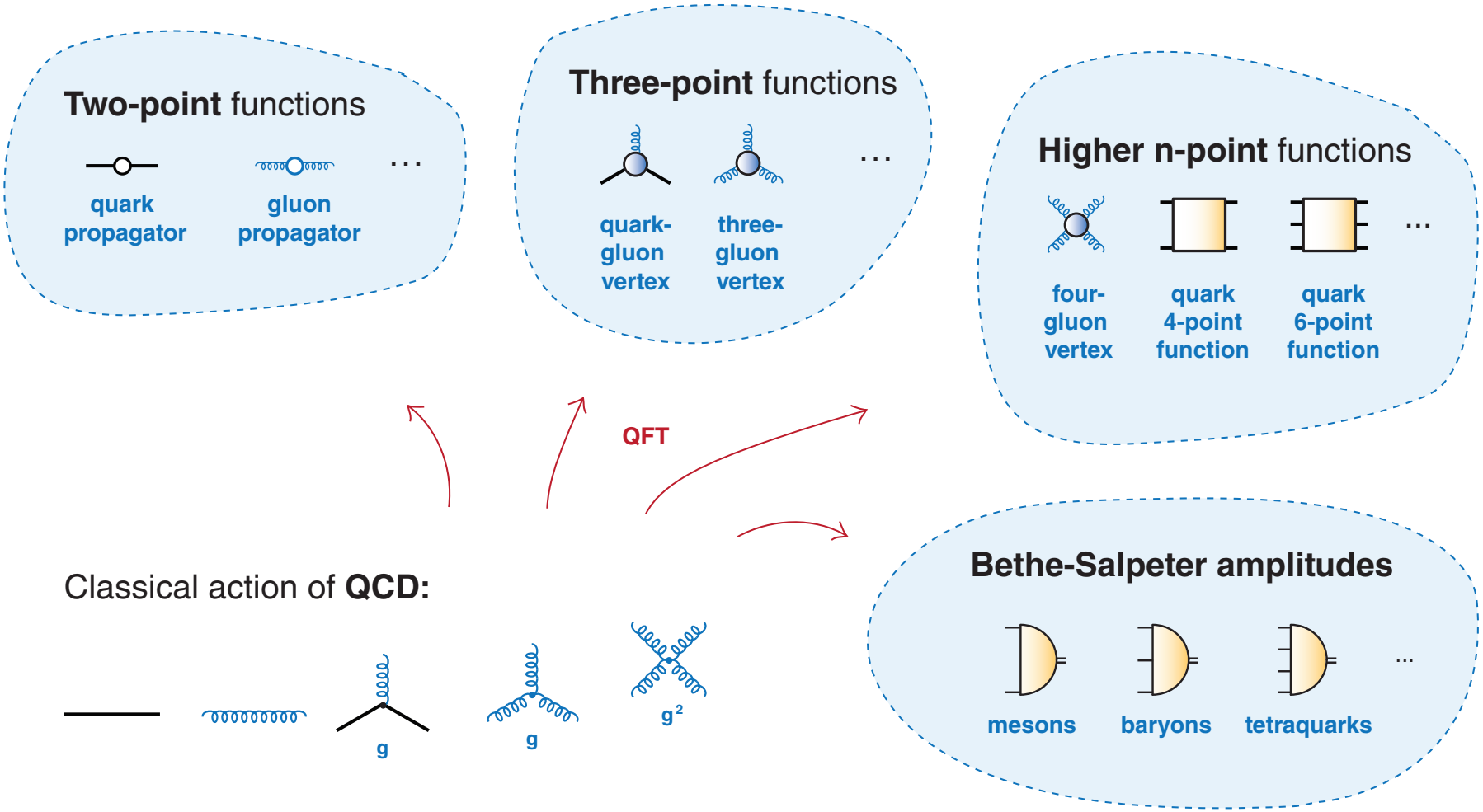
GE, "*Hadron physics with functional methods*",
arXiv: 2503.10397 (for Encyclopedia of Particle Physics)

Burkert, GE, Klempt, "*The impact of* γN *and* γ*N *interactions on our understanding of nucleon excitations*", arXiv:2506.16482

GE, Sanchis-Alepuz, Williams, Alkofer, Fischer,
"*Baryons as relativistic three-quark bound states*",
Prog. Part. Nucl. Phys. 91 (2016), arXiv: 1606.09602

Barabanov, Bedolla, Brooks, Cates, Chen et al.,
"*Diquark correlations in hadron physics: Origin, impact and evidence*", Prog. Part. Nucl. Phys. 116 (2021),
arXiv: 2008.07630

Correlation functions



Correlation functions

Two-point functions



Three-point functions



Higher n-point functions



Information is contained in Lorentz-invariant
dressing functions:

$$G^{\mu\nu\ldots}_{\alpha\beta\ldots}(p_1,\ldots p_n) = \sum_{i=1}^{N} f_i(p_1^2, p_2^2, \ldots)\, \tau_i(p_1,\ldots p_n)^{\mu\nu\ldots}_{\alpha\beta\ldots}$$

Kinematics? **Tensor bases?**

What can we learn from **symmetries?**

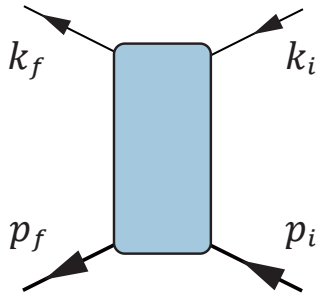
- Lorentz & parity invariance
- Crossing symmetry, charge conjugation
- Gauge invariance

Bethe-Salpeter amplitudes



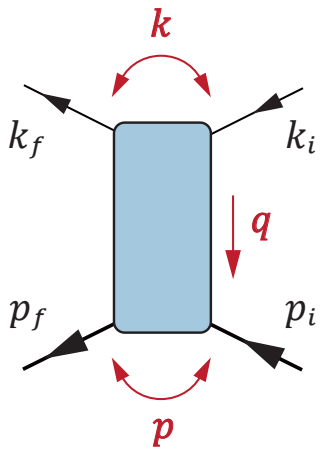
1. **Example: 2→2 scattering**
2. **Euclidean metric**
3. **QCD's n-point functions**
4. **Tensor bases**
5. **Kinematics**

2→2 scattering (1)



- **Scalar** ⇒ only one function $\Gamma(p_i, k_i, p_f, k_f)$
- 4 momenta + momentum conservation ⇒ **3 independent momenta**

2→2 scattering (1)



- **Scalar** ⇒ only one function $\Gamma(p_i, k_i, p_f, k_f)$
- 4 momenta + momentum conservation ⇒ **3 independent momenta,** define:

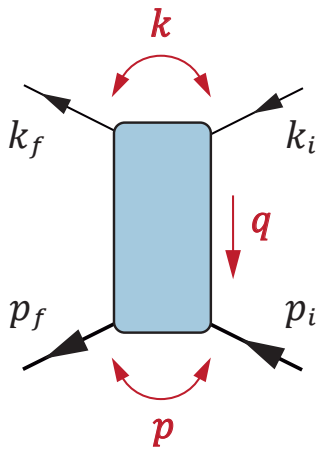
$$p = \frac{p_i + p_f}{2}\,, \qquad k = \frac{k_i + k_f}{2}\,, \qquad q = p_f - p_i = k_i - k_f$$

$$\Rightarrow \quad p_i = p - \frac{q}{2} \qquad k_i = k + \frac{q}{2}$$
$$p_f = p + \frac{q}{2} \qquad k_f = k - \frac{q}{2}$$

$$\Rightarrow \quad p_i + k_i = p_f + k_f$$

✓

2→2 scattering (1)



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- 3 momenta ⇒ **6 Lorentz invariants:**

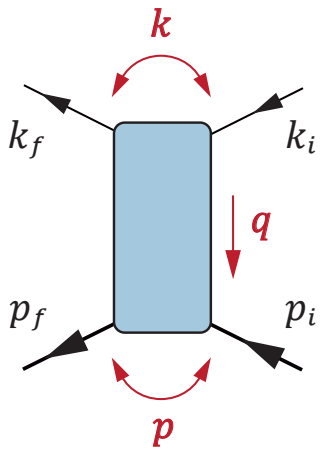
$$p^2, \quad q^2, \quad k^2, \quad p\cdot q, \quad k\cdot q, \quad p\cdot k$$

- Assume all particles are **onshell:** $p_i^2 = p_f^2 = M^2, \;\; k_i^2 = k_f^2 = m^2$

$$\left.\begin{aligned} p_i^2 &= p^2 + \tfrac{q^2}{4} - p\cdot q = M^2 \\ p_f^2 &= p^2 + \tfrac{q^2}{4} + p\cdot q = M^2 \end{aligned}\right\} \quad p\cdot q = 0, \quad p^2 = M^2 - \tfrac{q^2}{4}$$

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2→2 scattering (1)



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- 4 momenta + momentum conservation ⇒ **3 independent momenta,** define:

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$$\Rightarrow \quad p_i + k_i = p_f + k_f$$

- 3 momenta ⇒ **6 Lorentz invariants:**

$$\cancel{p^2}, \quad q^2, \quad \cancel{k^2}, \quad \cancel{p\cdot q}, \quad \cancel{k\cdot q}, \quad p\cdot k$$

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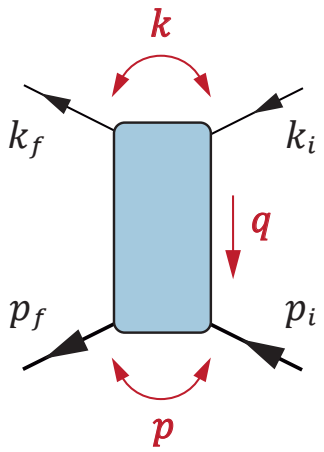
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⇒ **2 variables** left:

$$\tau = -\frac{q^2}{4M^2}$$
$$\lambda = \frac{p\cdot k}{M^2}$$

2→2 scattering (2)

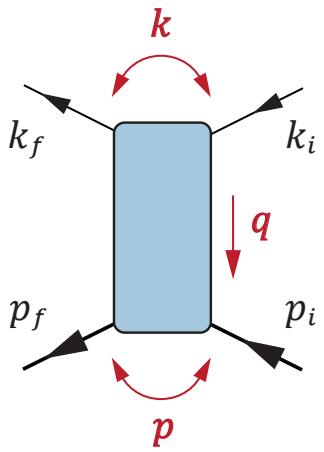


- 3 independent momenta, 2 Lorentz invariants:

$$p_i = p - \frac{q}{2} \qquad k_i = k + \frac{q}{2} \qquad \tau = -\frac{q^2}{4M^2} \qquad p \cdot q = 0, \quad p^2 = M^2 - \frac{q^2}{4}$$

$$p_f = p + \frac{q}{2} \qquad k_f = k - \frac{q}{2} \qquad \lambda = \frac{p \cdot k}{M^2} \qquad k \cdot q = 0, \quad k^2 = m^2 - \frac{q^2}{4}$$

2→2 scattering (2)



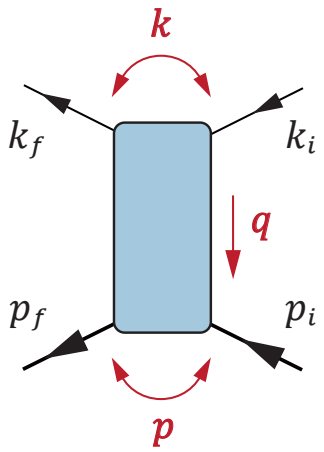
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- Define

$$\varepsilon = \frac{M^2 - m^2}{2M^2} \quad \Rightarrow \quad m^2 = M^2(1 - 2\varepsilon)$$
$$\Rightarrow \quad p^2 + k^2 = M^2 + m^2 - \frac{q^2}{2} = 2M^2(1 + \tau - \varepsilon)$$

2→2 scattering (2)



- 3 independent momenta, 2 Lorentz invariants:

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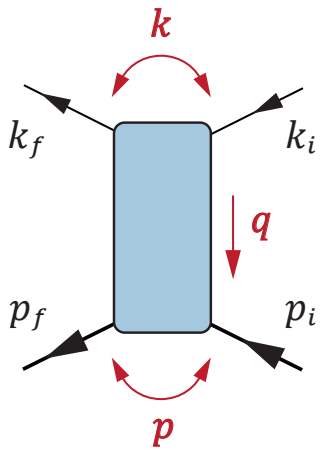
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- **Mandelstam variables:**

$$s = (p_i + k_i)^2 = (p_f + k_f)^2 = (p+k)^2 = p^2 + k^2 + 2p\cdot k = 2M^2(1+\tau+\lambda-\varepsilon)$$
$$u = (p_i - k_f)^2 = (p_f - k_i)^2 = (p-k)^2 = p^2 + k^2 - 2p\cdot k = 2M^2(1+\tau-\lambda-\varepsilon)$$
$$t = (p_f - p_i)^2 = (k_i - k_f)^2 = q^2 = -4M^2\tau$$

$$\Rightarrow \quad s + t + u = 4M^2(1-\varepsilon) = 2M^2 + 2m^2 \quad \checkmark$$

2→2 scattering (2)



- 3 independent momenta, 2 Lorentz invariants:

$$p_i = p - \tfrac{q}{2} \qquad k_i = k + \tfrac{q}{2} \qquad \tau = -\tfrac{q^2}{4M^2} \qquad p\cdot q = 0, \quad p^2 = M^2 - \tfrac{q^2}{4}$$
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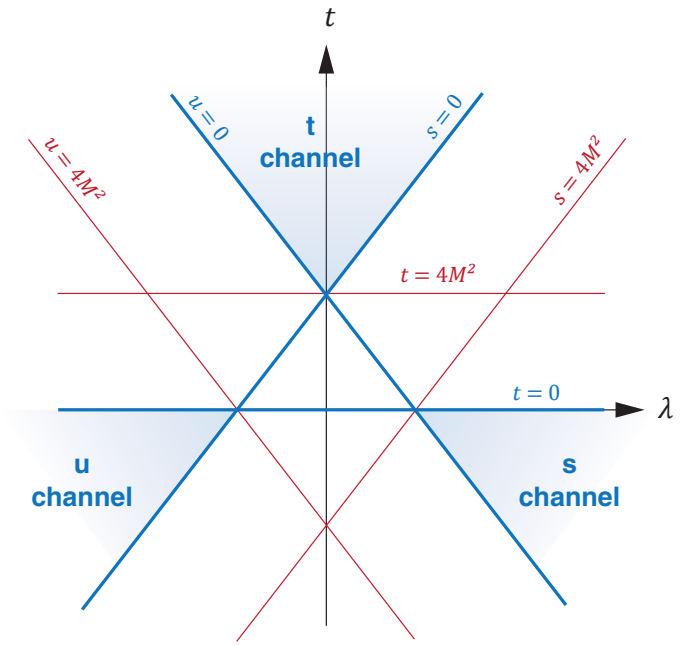
$$\Rightarrow \quad s + t + u = 4M^2(1-\varepsilon) = 2M^2 + 2m^2 \quad \checkmark$$

$$\Rightarrow \quad \Gamma(p_i, k_i, p_f, k_f) = f(s,t,u) = f(\lambda, \tau)$$

$$\Rightarrow \quad \textbf{Momentum transfer:} \quad \tau = -\frac{t}{4M^2} = \frac{s+u}{4M^2} - 1 + \varepsilon$$

Crossing variable: $\lambda = \frac{s-u}{4M^2}$

Mandelstam plane

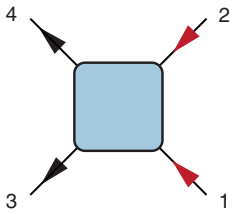


$$\tau = -\frac{t}{4M^2} = \frac{s+u}{4M^2} - 1 + \varepsilon$$

$$\lambda = \frac{s-u}{4M^2}$$

here: identical masses ($\varepsilon = 0$)

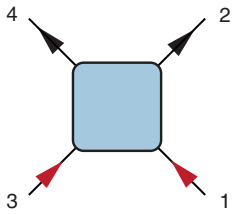
- Different processes described by same amplitude, but with different physical domains:



s channel:

$1 + 2 \to 3 + 4$

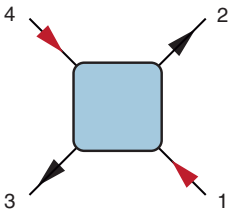
$N\pi^+ \to N\pi^+$



t channel:

$1 + \bar{3} \to \bar{2} + 4$

$N\bar{N} \to \pi^-\pi^+$



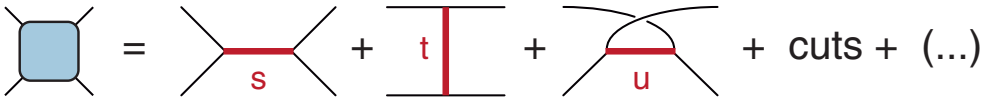
u channel:

$1 + \bar{4} \to \bar{2} + 3$

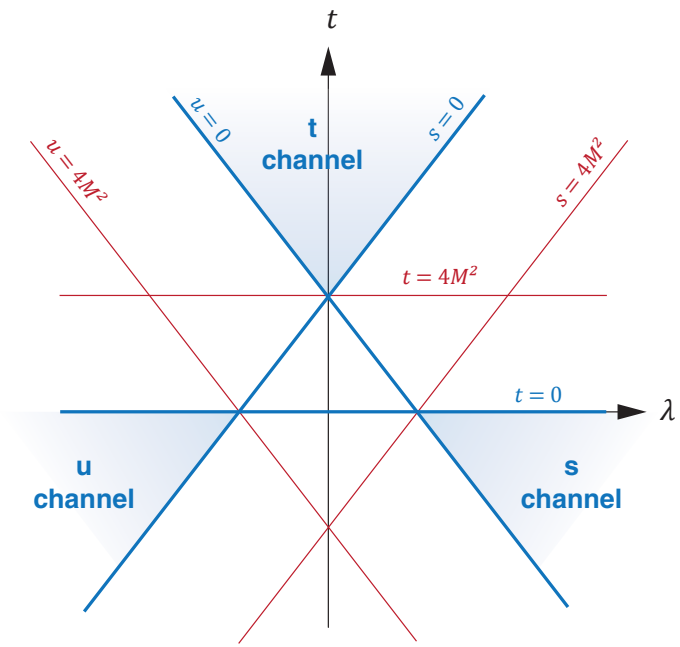
$N\pi^- \to \pi^- N$

e.g. $N\pi$ scattering

- Dressing functions are determined by (all) their singularities:



Mandelstam plane



$$\tau = -\frac{t}{4M^2} = \frac{s+u}{4M^2} - 1 + \varepsilon$$

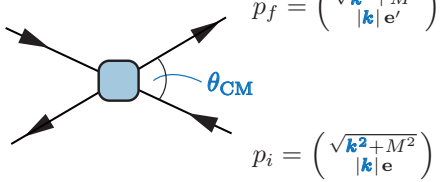
$$\lambda = \frac{s-u}{4M^2}$$

Amplitudes are **Lorentz-invariant,**
no need to go into specific frame
unless to relate with experiments:

- CM Frame:

$$\lambda = \frac{s-u}{M^2} = \frac{1}{M^2}\left[\frac{\boldsymbol{k}^2}{2}(1+\cos\theta_{\rm CM}) + \sqrt{\boldsymbol{k}^2+M^2}\sqrt{\boldsymbol{k}^2+m^2}\right]$$

$$\tau = \frac{\boldsymbol{k}^2}{2M^2}(1-\cos\theta_{\rm CM})$$

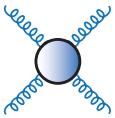


- Lab Frame:

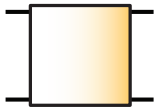
$$\left.\begin{aligned}\lambda &= \ldots\\ \tau &= \ldots\end{aligned}\right\} \Leftrightarrow \boldsymbol{k}_{\rm lab}\,,\ \theta_{\rm lab}$$

λ and τ are the same in any frame!

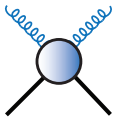
But ...



$\Gamma^{\mu\nu\rho\sigma}$



$\Gamma_{\alpha\beta\gamma\delta}$



$\Gamma^{\mu\nu}_{\alpha\beta}$

- Amplitudes are offshell:
  - → complicated **kinematic phase space** (here: 6 variables)
  - → how to find most convenient set of Lorentz invariants?

- Amplitudes have Dirac and/or Lorentz structure
  - → complicated **tensor bases**
  - → many **dressing functions**
  - → complicates problem both algebraically and numerically
  - → use **symmetries** to find efficient tensor bases:
    gauge invariance & permutation symmetries
    (crossing symmetry, charge conjugation)

After all that, we can
talk about **dynamics!**

1. Example: 2→2 scattering
2. **Euclidean metric**
3. QCD's n-point functions
4. Tensor bases
5. Kinematics

Euclidean conventions

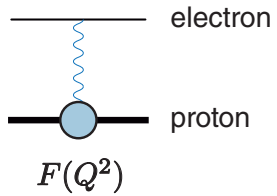
Why Euclidean?

- We're often dealing with **spacelike** momenta (virtual particles)
- **Loop momenta** are spacelike
- Minkowski metric (+, –, –, –) is inconvenient: upper & lower indices, cumbersome to write code
- QFT is already "Euclidean":
  To make QFT well-defined, one needs **imaginary-time boundary conditions:**

$$\int d^4x = \int d^3x \int_{-\infty(1-i\epsilon)}^{\infty(1-i\epsilon)} dx_0 \quad \Leftrightarrow \quad \int d^4p = \int d^3p \int_{-\infty(1+i\epsilon)}^{\infty(1+i\epsilon)} dp_0$$

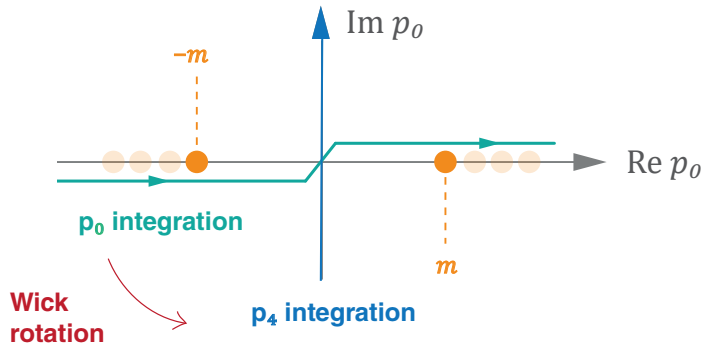
As long as amplitudes fall off fast enough at complex infinity, this is equivalent to Euclidean QFT!

$$\tau = -\frac{q^2}{4M^2} = \frac{Q^2}{4M^2}$$



Corresponds to $i\epsilon$ prescription in Feynman propagator:

$$\int_{-\infty}^{\infty} dp_0 \, \frac{1}{p^2 - m^2 + i\varepsilon} \quad \Rightarrow \quad p_0^2 = \boldsymbol{p}^2 + m^2 - i\epsilon$$



Euclidean conventions

So, let's employ a **Euclidean metric** (+, +, +, +):

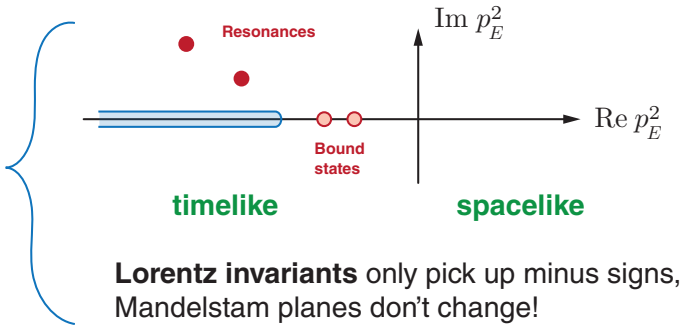
$$a_E^\mu = \begin{bmatrix} \boldsymbol{a} \\ i a_0 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} a_E \cdot b_E &= -a \cdot b \\ a_E^2 &= -a^2 \end{aligned}$$

To preserve meaning of slash $\not{a} = a^0\gamma^0 - \boldsymbol{a}\cdot\boldsymbol{\gamma}$, we must redefine γ matrices:

$$i\gamma_E^\mu = \begin{bmatrix} \boldsymbol{\gamma} \\ i\gamma_0 \end{bmatrix}, \quad \gamma_E^5 = \gamma^5 \quad \Rightarrow \quad \begin{aligned} \not{a}_E &= a_E \cdot \gamma_E = i\not{a} & (\gamma_E^i)^2 &= 1 \\ \{\gamma_E^\mu, \gamma_E^\nu\} &= 2\delta^{\mu\nu} & \gamma_E^\mu &= (\gamma_E^\mu)^\dagger \end{aligned}$$

⇒ Euclidean action is non-negative, defines probability measure:

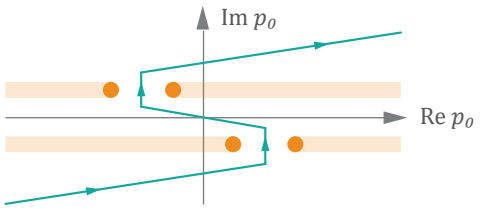
$$e^{iS} = \exp\left[i\int d^4x\, \bar\psi\,(i\not\partial - m)\,\psi\right] = \exp\left[-\int d^4x_E\, \bar\psi\,(\not\partial_E + m)\,\psi\right] = e^{-S_E}$$



Lorentz invariants only pick up minus signs, Mandelstam planes don't change!

Minkowski	Euclidean
$a \cdot b$	$-a \cdot b$
a^μ	a^μ
γ^μ	$i\gamma^\mu$
γ_5	γ_5
$\not{a}$	$-i\not{a}$
$g^{\mu\nu}$	$-\delta^{\mu\nu}$
$a^\mu b^\nu$	$a^\mu b^\nu$
$[\gamma^\mu, \gamma^\nu]$	$-[\gamma^\mu, \gamma^\nu]$
$[\gamma^\mu, \not{a}]$	$[\gamma^\mu, \not{a}]$
$[\gamma^\mu, \gamma^\nu, \not{a}]$	$i[\gamma^\mu, \gamma^\nu, \not{a}]$
$[\not{a}, \not{b}]$	$-[\not{a}, \not{b}]$
$\varepsilon^{\mu\nu\rho\alpha} a_\alpha$	$i\varepsilon^{\mu\nu\rho\alpha} a^\alpha$
$\varepsilon^{\mu\nu\alpha\beta} a_\alpha\, b_\beta$	$-i\varepsilon^{\mu\nu\alpha\beta} a^\alpha b^\beta$
$\varepsilon^{\mu\alpha\beta\gamma} a_\alpha\, b_\beta\, c_\gamma$	$i\varepsilon^{\mu\alpha\beta\gamma} a^\alpha b^\beta c^\gamma$
$\varepsilon^{\mu\nu\alpha\beta} a_\alpha \gamma_\beta$	$-\varepsilon^{\mu\nu\alpha\beta} a^\alpha \gamma^\beta$

Disclaimer: This does not mean you can simply close your eyes when computing loop integrals... if there are cuts in the way, you must still deform the integration contour (also in Euclidean)!



Euclidean conventions

From now on, I will use Euclidean conventions:

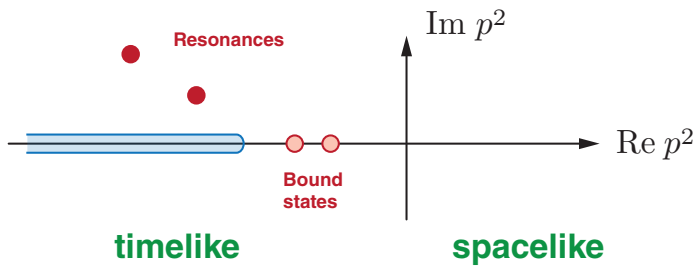
- No distinction between upper & lower indices
- $p^2 > 0$ means spacelike, $p^2 < 0$ timelike
- For an onshell momentum in the **rest frame,** we write

$$P = \begin{bmatrix} 0 \\ 0 \\ 0 \\ iM \end{bmatrix} \quad \Rightarrow \quad P^2 = -M^2$$

- For a general four momentum (e.g. **loop momenta**)

$$p = \sqrt{p^2} \begin{bmatrix} \sqrt{1-z^2}\sqrt{1-y^2}\sin\psi \\ \sqrt{1-z^2}\sqrt{1-y^2}\cos\psi \\ \sqrt{1-z^2}\, y \\ z \end{bmatrix}$$

- Easy to code!

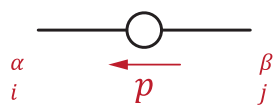


When people say
"We need to go to Minkowski space",
what they usually mean is:
*We need to go to the **timelike region**!*

1. Example: 2→2 scattering
2. Euclidean metric
3. **QCD's n-point functions**
4. Tensor bases
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Correlation functions in QCD

- **Quark propagator**



$$S(p) = \frac{1}{A(p^2)} \frac{-i\not{p} + M(p^2)}{p^2 + M(p^2)^2} \delta_{ij}$$

$$S(p)^{-1} = A(p^2)(i\not{p} + M(p^2))\,\delta_{ij}$$

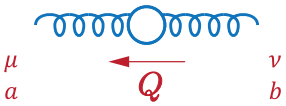
quark mass function

color

1 momentum,
2 Dirac indices ⇒ 1, $\not{p}$

$$S_{\alpha\beta}(p) = \sum_{i=1}^{2} f_i(p^2)\,\tau_i(p)_{\alpha\beta}$$

- **Gluon propagator**



$$D^{\mu\nu}(Q) = \frac{1}{Q^2}\left(Z(Q^2)\,T_Q^{\mu\nu} + \xi L_Q^{\mu\nu}\right)\delta_{ab}$$

gluon dressing

$$T_Q^{\mu\nu} = \delta^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2}$$

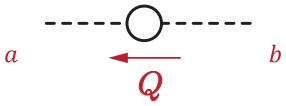
$$L_Q^{\mu\nu} = \frac{Q^\mu Q^\nu}{Q^2}$$

transverse & longitudinal projectors

1 momentum,
2 Lorentz indices ⇒ $\delta^{\mu\nu}$, $Q^\mu Q^\nu$

$$D^{\mu\nu}(Q) = \sum_{i=1}^{2} f_i(Q^2)\,\tau_i(Q)^{\mu\nu}$$

- **Ghost propagator**

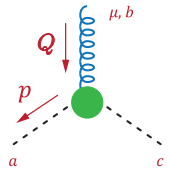


$$D_G(Q) = -\frac{G(Q^2)}{Q^2}\,\delta_{ab}$$

if (+) here, then also (+) in ghost-gluon vertex

Correlation functions in QCD

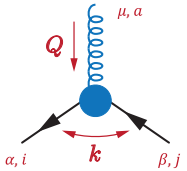
- **Ghost-gluon vertex**



2 momenta,
1 Lorentz index $\Rightarrow$ p^μ, Q^μ

$$\Gamma^\mu_{\rm gh}(p,Q) = -ig f_{abc}\left[\left(1 + A(p^2, p\cdot Q, Q^2)\right) p^\mu + B(p^2, p\cdot Q, Q^2)\, Q^\mu\right]$$

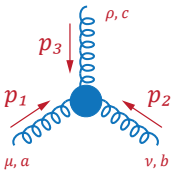
- **Quark-gluon vertex**



2 momenta,
1 Lorentz index,
2 Dirac indices

$$\Gamma^\mu_{\rm q}(k,Q) = ig\,\mathrm{t}_a \sum_{i=1}^{12}{}^{!}\, f_i(k^2, k\cdot Q, Q^2)\,\tau_i^\mu(k,Q)$$

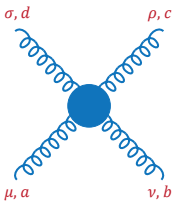
- **Three-gluon vertex**



2 momenta,
3 Lorentz indices

$$\Gamma^{\mu\nu\rho}_{\rm 3g}(p_1,p_2,p_3) = ig f_{abc} \sum_{i=1}^{14}{}^{!}\, f_i(p_1^2, p_2^2, p_3^2)\,\tau_i^{\mu\nu\rho}(p_1,p_2,p_3)$$

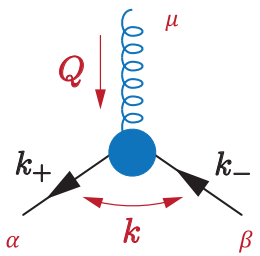
- **Four-gluon vertex**



3 momenta,
4 Lorentz indices,
5 color tensors

$$\Gamma^{\mu\nu\rho\sigma}_{\rm 4g}(p_1,p_2,p_3,p_4) = -g^2 \sum_{i=1}^{136}\sum_{j=1}^{5}{}^{!!}\, f_{ij}(\dots)\,\tau_i^{\mu\nu\rho\sigma}(p_1,p_2,p_3,p_4)\,\tau^{(j)}_{abcd}$$

Fermion-vector vertex



Let's consider a **general** fermion-vector vertex:

$$\Gamma^\mu(k,Q) = i \sum_{i=1}^{12} f_i(k^2, k\cdot Q, Q^2)\, \tau_i^\mu(k,Q)$$

quark-gluon vertex,
quark-photon vertex,
nucleon-photon vertex, ...

2 independent momenta: k_+, k_- or (simpler in practice) k, Q

$$k_\pm = k \pm \frac{Q}{2} \quad \Leftrightarrow \quad k = \frac{k_+ + k_-}{2}\,, \quad Q = k_+ - k_-$$

12 linearly independent tensors:

$$\begin{array}{cccc} \gamma^\mu & \gamma^\mu \not{k} & \gamma^\mu \not{Q} & \gamma^\mu \not{k} \not{Q} \\ k^\mu & k^\mu \not{k} & k^\mu \not{Q} & k^\mu \not{k} \not{Q} \\ Q^\mu & Q^\mu \not{k} & Q^\mu \not{Q} & Q^\mu \not{k} \not{Q} \end{array}$$

This is a complete basis.
But is it also the *best* choice?

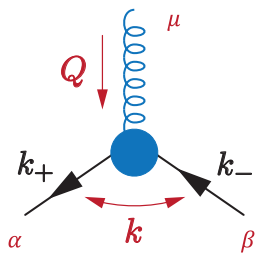
The vertex has a **charge-conjugation symmetry:**

$$\Gamma^\mu(k,Q) \stackrel{!}{=} - C\, \Gamma^\mu(-k,Q)^T\, C^T,$$

where $C = \gamma^4\gamma^2$ is the charge-conjugation matrix and satisfies

$$C^T = C^\dagger = C^{-1} = -C\,, \qquad C\gamma_5^T C^T = \gamma_5\,, \qquad C\gamma_\mu^T C^T = -\gamma_\mu\,.$$

Fermion-vector vertex (2)



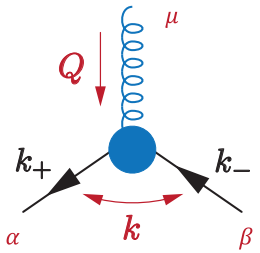
γ^μ	$\gamma^\mu \not{k}$	$\gamma^\mu \not{Q}$	$\gamma^\mu \not{k} \not{Q}$
k^μ	$k^\mu \not{k}$	$k^\mu \not{Q}$	$k^\mu \not{k} \not{Q}$
Q^μ	$Q^\mu \not{k}$	$Q^\mu \not{Q}$	$Q^\mu \not{k} \not{Q}$

$$\Gamma^\mu(k,Q) \stackrel{!}{=} -\, C\, \Gamma^\mu(-k,Q)^T\, C^T$$

$$C^T = C^\dagger = C^{-1} = -C\,,$$

$$C\gamma_5^T C^T = \gamma_5\,, \qquad C\gamma_\mu^T C^T = -\gamma_\mu$$

Fermion-vector vertex (2)



$$\begin{array}{cccc} \gamma^\mu & \gamma^\mu \not{k} & \gamma^\mu \not{Q} & \gamma^\mu \not{k} \not{Q} \\ k^\mu & k^\mu \not{k} & k^\mu \not{Q} & k^\mu \not{k} \not{Q} \\ Q^\mu & Q^\mu \not{k} & Q^\mu \not{Q} & Q^\mu \not{k} \not{Q} \end{array}$$

$$\Gamma^\mu(k,Q) \stackrel{!}{=} -C\,\Gamma^\mu(-k,Q)^T\,C^T$$

$$C^T = C^\dagger = C^{-1} = -C\,,$$

$$C\gamma_5^T C^T = \gamma_5\,, \qquad C\gamma_\mu^T C^T = -\gamma_\mu$$

But not all tensors satisfy this constraint:

$$\gamma_\mu \;\to\; -C\gamma_\mu^T C^T = \gamma_\mu \quad ✓$$

$$k^\mu \;\to\; -C(-k^\mu)C^T = k^\mu \quad ✓$$

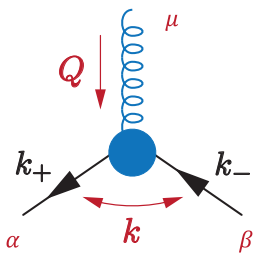
$$Q^\mu \;\to\; -C\,Q^\mu\,C^T = -\,Q^\mu \quad ✗$$

$$\gamma_\mu \not{Q} \;\to\; -C\,\not{Q}^T\gamma_\mu^T C^T = \not{Q}\,C\gamma_\mu^T C^T = -\not{Q}\,\gamma_\mu = \gamma_\mu \not{Q} - 2Q^\mu \quad ✗$$

$C^T C = 1$

leads to symmetry relations between dressing functions

Fermion-vector vertex (2)



$$\begin{array}{cccc} \gamma^\mu & \gamma^\mu \not{k} & \gamma^\mu \not{Q} & \gamma^\mu \not{k} \not{Q} \\ k^\mu & k^\mu \not{k} & k^\mu \not{Q} & k^\mu \not{k} \not{Q} \\ Q^\mu & Q^\mu \not{k} & Q^\mu \not{Q} & Q^\mu \not{k} \not{Q} \end{array}$$

$$\Gamma^\mu(k,Q) \stackrel{!}{=} -C\,\Gamma^\mu(-k,Q)^T\,C^T$$

$$C^T = C^\dagger = C^{-1} = -C\,,$$

$$C\gamma_5^T C^T = \gamma_5\,, \qquad C\gamma_\mu^T C^T = -\gamma_\mu$$

But not all tensors satisfy this constraint:

$$\gamma_\mu \;\to\; -C\gamma_\mu^T C^T = \gamma_\mu \quad ✓$$

$$k^\mu \;\to\; -C(-k^\mu)C^T = k^\mu \quad ✓$$

$$Q^\mu \;\to\; -C\,Q^\mu C^T = -\,Q^\mu \quad ✗$$

$$\gamma_\mu \not{Q} \;\to\; -C\,\not{Q}^T\gamma_\mu^T C^T = \not{Q}\,C\gamma_\mu^T C^T = -\not{Q}\,\gamma_\mu = \gamma_\mu \not{Q} - 2Q^\mu \quad ✗$$

$C^TC = 1$

leads to symmetry relations between dressing functions

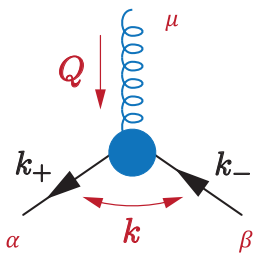
Simpler: implement symmetry directly for the basis elements

$$k\cdot Q\; Q^\mu \;\to\; +k\cdot Q\; C Q^\mu C^T = k\cdot Q\; Q^\mu \quad ✓$$

$$[\gamma_\mu, \not{Q}] \;\to\; -C\,[\not{Q}^T, \gamma_\mu^T]\,C^T = -[C\not{Q}^T C^T, C\gamma_\mu^T C^T] = -[\not{Q}, \gamma_\mu] = [\gamma_\mu, \not{Q}] \quad ✓$$

In short: use **commutators** & attach factors $k\cdot Q$ where necessary

Fermion-vector vertex (3)



Abbreviating $\omega = k \cdot Q$, we arrive at the new basis

γ^μ	$i\omega[\gamma^\mu, \not{k}]$	$i[\gamma^\mu, \not{Q}]$	$[\gamma^\mu, \not{k}, \not{Q}]$
ik^μ	$k^\mu \not{k}$	$\omega k^\mu \not{Q}$	$ik^\mu [\not{k}, \not{Q}]$
$i\omega Q^\mu$	$\omega Q^\mu \not{k}$	$Q^\mu \not{Q}$	$i\omega Q^\mu [\not{k}, \not{Q}]$

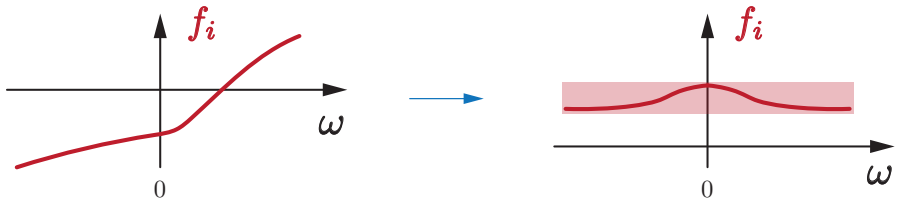
← Triple commutator is antisymmetric combination of A, B, C :
$[A, B, C] = [A, B]\, C + [B, C]\, A + [C, A]\, B$

← i factors make all dressing functions real

Now all tensors satisfy the constraint $\Gamma^\mu(k, Q) \stackrel{!}{=} - C\, \Gamma^\mu(-k, Q)^T\, C^T$ individually, so their dressing functions must be **even** in ω, can only depend on ω^2 :

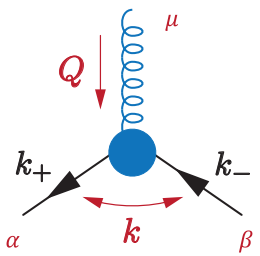
$$\Gamma^\mu(k, Q) = i \sum_{i=1}^{12} f_i(k^2, \omega^2, Q^2)\, \tau_i^\mu(k, Q)$$

This also simplifies their momentum dependence:



Mathematically speaking, we have arranged the tensors and dressing functions into **singlets** of the **permutation group** S_2 !

Fermion-vector vertex (4)



Amplitudes with gluon or photon legs are subject to **gauge invariance,** e.g. **Ward-Takahashi identity** (WTI) for quark-photon vertex:

$$Q^\mu \Gamma^\mu(k,Q) = S^{-1}(k_+) - S^{-1}(k_-)$$ → fixes **longitudinal** part of vertex

We could split the vertex into transverse and longitudinal part:

$$\Gamma^\mu(k,Q) = T_Q^{\mu\nu}\,\Gamma^\nu(k,Q) + L_Q^{\mu\nu}\,\Gamma^\nu(k,Q)$$
$$= \Gamma_\perp^\mu(k,Q) + \frac{Q^\mu}{Q^2}\left[S^{-1}(k_+) - S^{-1}(k_-)\right]$$

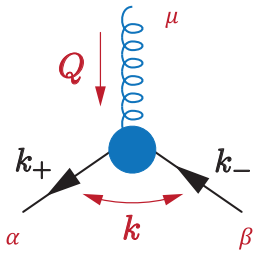
But this has a **kinematic singularity** at $Q^2 = 0$, violates **Ward identity:**

$$\Gamma^\mu(k,0) = \frac{dS(k)^{-1}}{dk^\mu}$$

⇒ Constructing tensor bases with T + L projectors is **dangerous** if we are interested in kinematic limits...

⇒ Must work out transverse part of vertex without introducing kinematic singularities

Fermion-vector vertex (5)



Simpler example: **scalar vertex**

$$\Gamma^\mu(k,Q) = c_1\, k^\mu + c_2\, \omega\, Q^\mu$$ → has correct permutation symmetry

Ward-Takahashi identity:

$$Q^\mu \Gamma^\mu(k,Q) = D^{-1}(k_+) - D^{-1}(k_-)$$

Let's first work out the transverse part:

$$Q^\mu \Gamma^\mu(k,Q) = \omega\,(c_1 + c_2\, Q^2) \stackrel{!}{=} 0 \quad \Rightarrow \quad c_1 = -c_2\, Q^2$$

Careful: don't use $c_2 = -\frac{c_1}{Q^2}$, which would introduce kinematic singularity!

$$\Rightarrow \quad \Gamma^\mu_\perp(k,Q) = -c_2\,(Q^2 k^\mu - \omega\, Q^\mu)$$
$$= -c_2\,\underbrace{(Q^2\delta^{\mu\nu} - Q^\mu Q^\nu)}_{=:\, t^{\mu\nu}_{QQ}}\, k^\nu$$

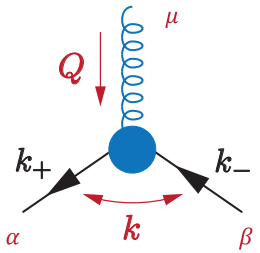
No $1/Q^2$ terms here!

Full vertex → we solved for c_1, so we must put back k^μ:

$$\Gamma^\mu(k,Q) = g_1\, k^\mu + f_1\, t^{\mu\nu}_{QQ}\, k^\nu$$

"minimal basis": no kinematic singularities, tensors are ordered with powers of momenta

Fermion-vector vertex (5)



Now we can work out the
Ward-Takahashi identity:

$$Q^\mu \Gamma^\mu(k,Q) = D^{-1}(k_+) - D^{-1}(k_-) \;=\; g_1\,\omega$$

Define

$$\Delta = \frac{D^{-1}(k_+) - D^{-1}(k_-)}{k_+^2 - k_-^2} \qquad k_\pm^2 = k^2 + \tfrac{Q^2}{4} \pm k\cdot Q \quad \Rightarrow \quad k_+^2 - k_-^2 = 2k\cdot Q = 2\omega$$

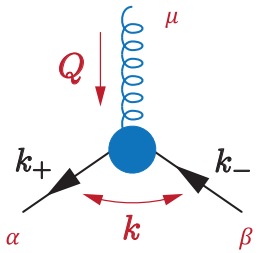
$$\Rightarrow\; g_1 = 2\Delta$$

Full vertex → we solved for c_1, so we must put back k^μ:

$$\Gamma^\mu(k,Q) = g_1\,k^\mu + f_1\,t^{\mu\nu}_{QQ}\,k^\nu$$

"minimal basis": no kinematic singularities,
tensors are ordered with powers of momenta

Fermion-vector vertex (6)



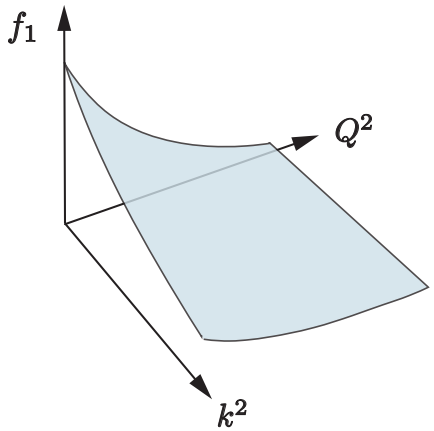
Final result for **scalar vertex:**

$$\Gamma^\mu(k,Q) = \; 2\Delta\, k^\mu + f_1\, t^{\mu\nu}_{QQ}\, k^\nu \qquad \Delta = \frac{D^{-1}(k_+) - D^{-1}(k_-)}{k_+^2 - k_-^2}$$

Satisfies WTI **and** Ward identity:

$$\Gamma^\mu(k,0) = 2\Delta\, k^\mu \;\; = \frac{dD^{-1}(k)}{dk^2}\, 2k^\mu = \frac{dD^{-1}(k)}{dk^\mu}$$ ✔

In this way, we eliminated all kinematic effects from tensor basis:
Dressing functions are free of kinematic constraints at $k = 0$ or $Q = 0$
(kinematically independent), no kinematic singularities or zeros:
their only singularities are physical poles and cuts!

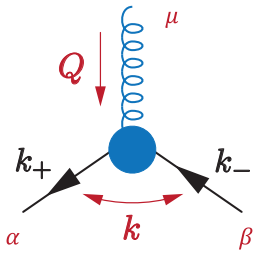


Also: tensors with higher momentum powers are suppressed at low momenta
⇒ ordering principle!

$$2\Delta\, k^\mu + f_1\, t^{\mu\nu}_{QQ}\, k^\nu$$

important — less important

Fermion-vector vertex (7)



The same can be done for **fermion-vector vertex:**

$$\Gamma^\mu(k,Q) = i\left[\sum_{j=1}^{4} g_j\, G_j^\mu + \sum_{j=1}^{8} f_j\, T_j^\mu\right]$$

Gauge part:
constrained by STI

γ^μ

$k^\mu \not{k}$

ik^μ

$\omega \frac{i}{2}[\gamma^\mu, \not{k}]$

Transverse part,
carries dynamics

$t^{\mu\nu}_{QQ}\,\gamma^\nu$

$\omega\, t^{\mu\nu}_{QQ}\,\frac{i}{2}[\gamma^\nu, \not{k}]$

$\frac{i}{2}[\gamma^\mu, \not{Q}]$

$\frac{1}{6}[\gamma^\mu, \not{k}, \not{Q}]$

$t^{\mu\nu}_{QQ}\, ik^\nu$

$t^{\mu\nu}_{QQ}\, k^\nu \not{k}$

$\omega\, t^{\mu\nu}_{Qk}\,\gamma^\nu$

$t^{\mu\nu}_{Qk}\,\frac{i}{2}[\gamma^\nu, \not{k}]$

$t^{\mu\nu}_{ab} = a\cdot b\,\delta^{\mu\nu} - b^\mu a^\nu$

$\Rightarrow\ a^\mu\, t^{\mu\nu}_{ab} = 0$

$t^{\mu\nu}_{ab}\, b^\nu = 0$

For quark-photon vertex: **Ball-Chiu vertex**

Ball, Chiu, PRD 22 (1980)

$$g_1 = \Sigma_A \qquad g_2 = 2\Delta_A \qquad g_3 = -2\Delta_B \qquad g_4 = 0$$

$$\Sigma_A = \frac{A(k_+) + A(k_-)}{2}$$

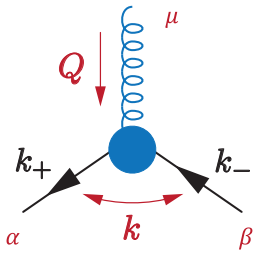
$$\Delta_A = \frac{A(k_+) - A(k_-)}{k_+^2 - k_-^2}$$

$$B(k^2) = A(k^2)M(k^2)$$

Everything else is **dynamics!** Quark-photon vertex can be calculated from BSE, f_j must have physical vector-meson poles at timelike momenta

Maris, Tandy, PRC 61 (2000)

Fermion-vector vertex (7)



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Gauge part: constrained by STI

Transverse part, carries dynamics

$t_{ab}^{\mu\nu} = a\cdot b\,\delta^{\mu\nu} - b^\mu a^\nu$

$\Rightarrow\ a^\mu\, t_{ab}^{\mu\nu} = 0$

$t_{ab}^{\mu\nu}\, b^\nu = 0$

lowest momentum powers

γ^μ	$t_{QQ}^{\mu\nu}\,\gamma^\nu$	$t_{QQ}^{\mu\nu}\, ik^\nu$
$k^\mu \not{k}$	$\omega\, t_{QQ}^{\mu\nu}\,\frac{i}{2}[\gamma^\nu, \not{k}]$	$t_{QQ}^{\mu\nu}\, k^\nu \not{k}$
ik^μ	$\frac{i}{2}[\gamma^\mu, \not{Q}]$	$\omega\, t_{Qk}^{\mu\nu}\,\gamma^\nu$
$\omega\,\frac{i}{2}[\gamma^\mu, \not{k}]$	$\frac{1}{6}[\gamma^\mu, \not{k}, \not{Q}]$	$t_{Qk}^{\mu\nu}\,\frac{i}{2}[\gamma^\nu, \not{k}]$

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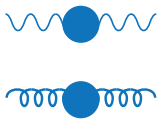
Fermion-vector vertex (8)

We have exploited all symmetries of the system:

- Permutation-group symmetries (here: charge conjugation)
- Gauge invariance

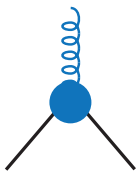
Procedure is general:

gluon self-energy, hadronic vacuum polarization



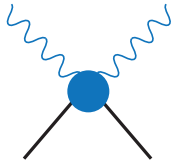
1 + 1

fermion-vector: quark-gluon vertex, quark-photon vertex



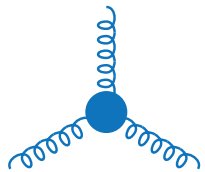
4 + 8

fermion-2-vector: nucleon Compton amplitude



14 + 18

3-gluon vertex



10 + 4

Gauge + Transverse

Bardeen, Tung, Phys. Rev. 173 (1968)
Tarrach, Nuovo Cim. A28 (1975)
Ball, Chiu, PRD 22 (1980)
Drechsel et al., PRC 55 (1997)
GE, Fischer, PRD 87 (2013)
GE, Ramalho, PRD 98 (2018)

Everything else is **dynamics!**

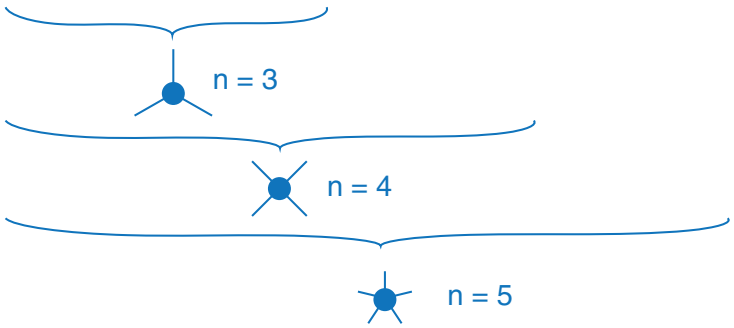
1. Example: 2→2 scattering
2. Euclidean metric
3. QCD's n-point functions
4. **Tensor bases**
5. Kinematics

Tensor bases

What is the **number of tensors** for a general n-point function?

GE, Fischer, PRD 87 (2013)
GE, Fischer, Heupel, PRD 92 (2015)
Wallbott, GE, Fischer, PRD 100 (2019)

$$\Gamma(p,q,k,l,\dots) \rightarrow p=\begin{bmatrix}0\\0\\0\\\bullet\end{bmatrix}, \quad q=\begin{bmatrix}0\\0\\\bullet\\\bullet\end{bmatrix}, \quad k=\begin{bmatrix}0\\\bullet\\\bullet\\\bullet\end{bmatrix}, \quad l=\begin{bmatrix}\bullet\\\bullet\\\bullet\\\bullet\end{bmatrix}, \quad \cdots$$

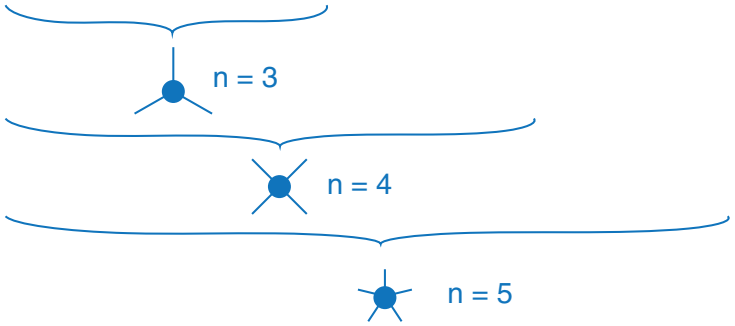


Tensor bases

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These are all Lorentz-covariant definitions, in principle we never need to go into a frame!

Construct orthonormal **unit vectors:**

$$n_4 = \hat{p} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix},\quad n_3 = \widehat{q_{\perp 4}} = \mathrm{Hat}\left[q-(q\cdot n_4)\,n_4\right] = \begin{bmatrix}0\\0\\1\\0\end{bmatrix},\quad n_2 = \widehat{k_{\perp 4,\perp 3}} = \mathrm{Hat}\left[k-(k\cdot n_4)\,n_4-(k\cdot n_3)\,n_3\right] = \begin{bmatrix}0\\1\\0\\0\end{bmatrix},$$

$$n_1 = \widehat{l_{\perp 4,\perp 3,\perp 2}} = \mathrm{Hat}\left[\dots\right] = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

... but then it stops: we are limited by **four dimensions.**

⇒ For arbitrary n-point functions, still at most **4 momenta** as building blocks!

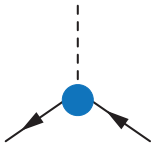
Tensor bases

Three-point functions:
two vectors n_3, n_4



$$\Gamma(p,q) \;\to\; p = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \bullet \end{bmatrix}, \quad q = \begin{bmatrix} 0 \\ 0 \\ \bullet \\ \bullet \end{bmatrix}$$

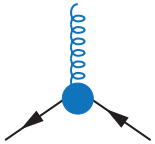
- Fermion-scalar vertex:



$$\{1,\ \not{n}_3,\ \not{n}_4,\ \not{n}_3\not{n}_4\} \quad \Rightarrow \quad \textbf{4} \text{ tensors}$$

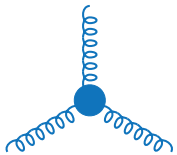
Same for fermion-pseudoscalar vertex (e.g., pion Bethe-Salpeter amplitude):
$\{1,\ \not{n}_3,\ \not{n}_4,\ \not{n}_3\not{n}_4\}\,\gamma_5$

- Fermion-vector vertex:



$$\{\gamma^\mu,\ n_3^\mu,\ n_4^\mu\} \times \{1,\ \not{n}_3,\ \not{n}_4,\ \not{n}_3\not{n}_4\} \quad \Rightarrow \quad \textbf{12} \text{ tensors}$$

- Three-gluon vertex:



$$\underbrace{n_i^\mu\, n_j^\nu\, n_k^\rho}_{2^3=8},\quad \underbrace{\delta^{\mu\nu}\, n_i^\rho}_{2},\quad \underbrace{\delta^{\nu\rho}\, n_i^\mu}_{2},\quad \underbrace{\delta^{\rho\mu}\, n_i^\nu}_{2} \quad \Rightarrow \quad \textbf{14} \text{ tensors}$$

Tensor bases

Four-point functions:
three vectors n_2, n_3, n_4



But with 3 vectors we can also do this:

$$v^\mu = \varepsilon^{\mu\alpha\beta\gamma}\, n_1^\alpha\, n_2^\beta\, n_3^\gamma = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow$$

this has opposite parity
⇒ we will need even numbers of v's

$$\begin{cases} \delta^{\mu\nu} = v^\mu\, v^\nu + \sum_{i=1}^{3} n_i^\mu\, n_i^\nu \\ \gamma^\mu = v^\mu\, \not{v} + \sum_{i=1}^{3} n_i^\mu\, \not{n}_i \\ \not{v} = \gamma_5\, \not{n}_2\, \not{n}_3\, \not{n}_4 \end{cases}$$

These relations hold in simple frame, but **Lorentz-covariant**
⇒ they are true in any frame!

⇒ we no longer need $\delta^{\mu\nu}$ and γ^μ in the basis construction, the n_i^μ and v^μ are enough!

Tensor bases

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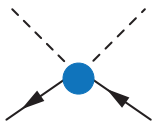
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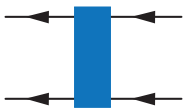
- Fermion-2-scalar vertex:



$$\Omega = \{1,\ \underbrace{\not{n}_i}_{3},\ \underbrace{\not{n}_i\, \not{n}_j}_{3},\ \underbrace{\not{n}_i\, \not{n}_j\, \not{n}_k}_{1}\},\quad i<j<k \qquad \Rightarrow \quad \textbf{8} \text{ tensors}$$

($\not{n}_i\, \not{n}_j\, \not{n}_k = \gamma_5\, \not{v}$)

- Fermion 4-point function:



$$\underbrace{\Omega_{\alpha\beta}\, \Omega_{\gamma\delta}}_{8 \times 8 = 64},\quad \underbrace{(\gamma_5\Omega)_{\alpha\beta}\, (\gamma_5\Omega)_{\gamma\delta}}_{8 \times 8 = 64} \qquad \Rightarrow \quad \textbf{128} \text{ tensors}$$

For quarks there are also 2 color tensors: 256 in total

Tensor bases

Four-point functions:
three vectors n_2, n_3, n_4



But with 3 vectors we can also do this:

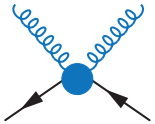
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These relations hold in simple frame, but **Lorentz-covariant**
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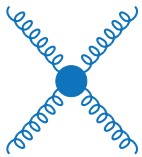
⇒ we no longer need $\delta^{\mu\nu}$ and γ^μ in the basis construction, the n_i^μ and v^μ are enough!

- Fermion-2-vector vertex:



$$\{\underbrace{v^\mu\, v^\nu}_{1},\ \underbrace{n_i^\mu\, n_j^\nu}_{9}\} \times \underbrace{\Omega}_{8},\quad \{\underbrace{v^\mu\, n_i^\nu}_{3},\ \underbrace{n_i^\mu\, v^\nu}_{3}\} \times \underbrace{\gamma_5\Omega}_{8} \quad \Rightarrow \quad \textbf{128} \text{ tensors}$$

- 4-gluon vertex:



$$\underbrace{v^\mu v^\nu v^\rho v^\sigma}_{1},\quad \underbrace{v^\mu v^\nu n_i^\rho\, n_j^\sigma}_{3^2 = 9,\ 6 \text{ permutations}},\quad \underbrace{n_i^\mu\, n_j^\nu\, n_k^\rho\, n_l^\sigma}_{3^4 = 81} \quad \Rightarrow \quad \textbf{136} \text{ tensors } (\times \text{ color})$$

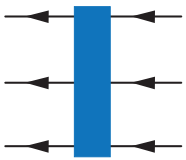
Tensor bases

> Four-point functions: four vectors n_1, n_2, n_3, n_4

$$\Rightarrow \quad \begin{cases} \delta^{\mu\nu} = \sum_{i=1}^{4} n_i^\mu \, n_i^\nu \\ \gamma^\mu = \sum_{i=1}^{4} n_i^\mu \, \not{n}_i \\ \not{n}_1 \not{n}_2 \not{n}_3 \not{n}_4 = -\varepsilon \gamma_5 \qquad \varepsilon = \varepsilon^{\mu\nu\rho\sigma} \, n_1^\mu \, n_2^\nu \, n_3^\rho \, n_4^\sigma \end{cases} \qquad \Rightarrow \gamma_5 \text{ becomes redundant too}$$

$$\Rightarrow \quad \Omega = \{ \underbrace{1}_{1}, \underbrace{\not{n}_i}_{4}, \underbrace{\not{n}_i \not{n}_j}_{6}, \underbrace{\not{n}_i \not{n}_j \not{n}_k}_{4}, \underbrace{\not{n}_i \not{n}_j \not{n}_k \not{n}_l}_{1} \}, \qquad i<j<k<l \qquad \Rightarrow \quad \textbf{16} \text{ tensors}$$

- Fermion 6-point function:



$$\underbrace{\Omega_{\alpha\beta} \, \Omega_{\gamma\delta} \, \Omega_{\lambda\tau}}_{16^3} \quad \Rightarrow \quad \textbf{4096} \text{ tensors} \qquad (\times \text{ color})$$

1. Example: 2→2 scattering
2. Euclidean metric
3. QCD's n-point functions
4. Tensor bases
5. **Kinematics**

Lorentz invariants

$$G^{\mu\nu\dots}_{\alpha\beta\dots}(p_1,\dots p_n) = \sum_{i=1}^{N} f_i(p_1^2, p_2^2, \dots)\, \tau_i(p_1,\dots p_n)^{\mu\nu\dots}_{\alpha\beta\dots}$$



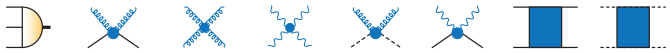
What is the **number of Lorentz invariants** for a general n-point function?

Lorentz invariants

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What is the **number of Lorentz invariants** for a general n-point function?
It's the number of entries in the vectors:

$$\Gamma(p,q,k,l,\ldots) \;\rightarrow\; p = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \bullet \end{bmatrix}, \quad q = \begin{bmatrix} 0 \\ 0 \\ \bullet \\ \bullet \end{bmatrix}, \quad k = \begin{bmatrix} 0 \\ \bullet \\ \bullet \\ \bullet \end{bmatrix}, \quad l = \begin{bmatrix} \bullet \\ \bullet \\ \bullet \\ \bullet \end{bmatrix}, \quad \ldots$$

n	# Lorentz invariants	Examples
2	**1**	
3	**3**	
4	**6**	
5	**10**	
6	**14**	
7	**18**	
n ≥ 4	**4n – 10**	

apart from possible
onshell constraints:
$P^2 = -m^2$, etc.

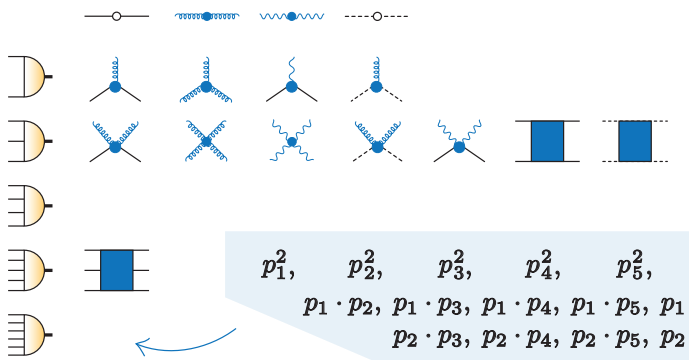
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n ≥ 4	**4n – 10**	



$p_1^2,\ p_2^2,\ p_3^2,\ p_4^2,\ p_5^2,\ p_6^2$ ... 6
$p_1\cdot p_2,\ p_1\cdot p_3,\ p_1\cdot p_4,\ p_1\cdot p_5,\ p_1\cdot p_6$ ... 5
$p_2\cdot p_3,\ p_2\cdot p_4,\ p_2\cdot p_5,\ p_2\cdot p_6$ ... 4
$p_3\cdot p_4,\ p_3\cdot p_5,\ p_3\cdot p_6$ ... 3
$p_4\cdot p_5,\ p_4\cdot p_6$ ... 2
$p_5\cdot p_6$ ... 1

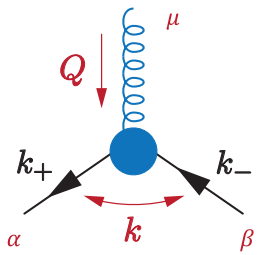
= 21, not 18...
even the
Lorentz-invariants
become redundant!

apart from possible
onshell constraints:
$P^2 = -m^2$, etc.

Lorentz invariants

$$G^{\mu\nu\ldots}_{\alpha\beta\ldots}(p_1,\ldots p_n) = \sum_{i=1}^{N} f_i(p_1^2, p_2^2, \ldots)\, \tau_i(p_1,\ldots p_n)^{\mu\nu\ldots}_{\alpha\beta\ldots}$$

What is the most **convenient** set of Lorentz invariants?



E.g. fermion vector vertex: kinematics invariant under $k_+ \leftrightarrow -k_-$

$k = \frac{k_+ + k_-}{2}$ → antisymmetric

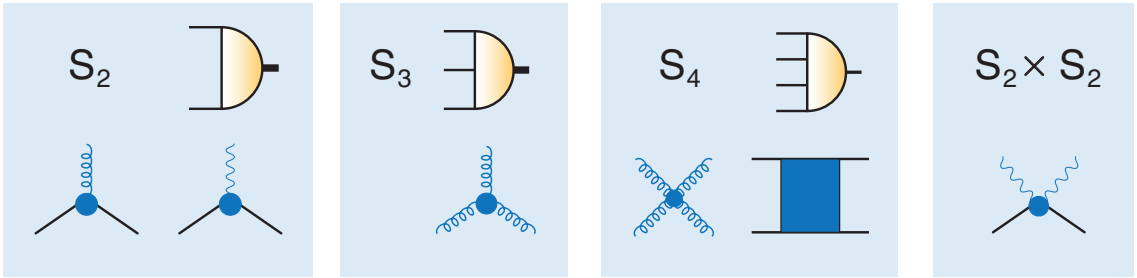
$Q = k_+ - k_-$ → symmetric

⇒ k^2, Q^2 are symmetric, $k \cdot Q$ is antisymmetric

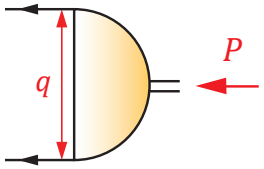
⇒ These variables are **multiplets** under permutation group S_2

Use permutation-group symmetries!

→



Example: diquark amplitude



$$\Gamma(p,P) = \underbrace{\sum_{i=1}^{4} f_i(q^2, q\cdot P, P^2)\,\tau_i(q,P)}_{\mathcal{A}} \;\overset{!}{\Leftarrow}\; \underbrace{\otimes \text{ Flavor}}_{\mathcal{A}:\; ud-du} \;\overset{!}{\Leftarrow}\; \underbrace{\otimes \text{ Color}}_{\mathcal{A}:\; \varepsilon_{ABC}} \;\overset{!}{=}\; \mathcal{A}$$

Pauli principle

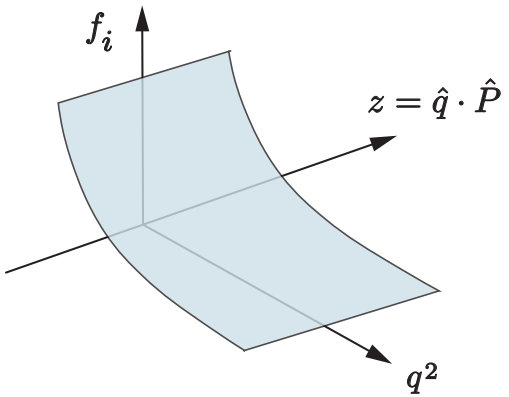
$$\mathcal{S}: \quad uu, \quad ud+du, \quad dd$$

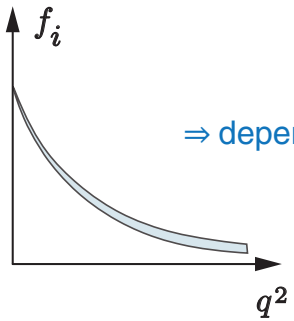
Scalar diquarks: $\tau_i = \{1, \dots\}\,\gamma_5 C$ ... $\mathcal{A}$ $\quad (\gamma_5 C)^T = C^T \gamma_5^T = -\gamma_5 C$

Axialvector diquarks: $\tau_i^\mu = \{\gamma^\mu, \dots\}\,C$ ... $\mathcal{S}$ $\quad (\gamma_\mu C)^T = C^T \gamma_\mu^T = +\gamma_\mu C$

⇒ Scalar diquarks have **isospin I = 0**, axialvector diquarks **isospin I = 1**

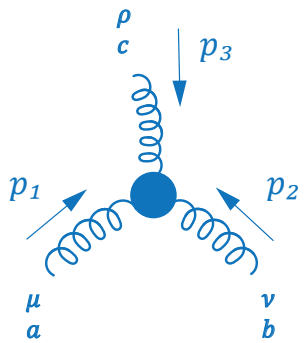
can construct remaining tensors to have definite symmetry (commutators, factors $q\cdot P$), such that f_i are singlets





⇒ depend mainly on q^2 !

Three-gluon vertex



$$\Gamma^{\mu\nu\rho}_{3g}(p_1,p_2,p_3) = ig \underbrace{\sum_{i=1}^{14} f_i(p_1^2,p_2^2,p_3^2)\, \tau_i^{\mu\nu\rho}(p_1,p_2,p_3)}_{\mathcal{A}} \underbrace{f_{abc}}_{\mathcal{A}} \overset{!}{=} \mathcal{S}$$

$\mathcal{A} \overset{!}{\Leftarrow} \mathcal{A}$

Bose symmetry

How can we make the symmetry manifest?

(1) Arrange kinematic variables in multiplets of permutation group S_3

(2) Arrange tensors in multiplets of permutation group S_3: then all tensors & dressing functions have definite symmetry

(3) Make tensors fully antisymmetric ⇒ then all dressing functions are fully symmetric

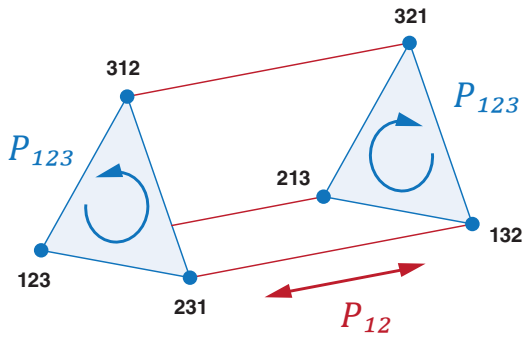
This requires some algebra... let's focus on (1)

GE, Williams, Alkofer, Vujinovic, PRD 89 (2014)

Permutation group S_3

Permutation group S_3 consists of 6 group elements, can be reconstructed from **transposition** P_{12} $(1 \leftrightarrow 2)$ and **cyclic permutation** P_{123} $(1 \to 2,\ 2 \to 3,\ 3 \to 1)$:

$$1\ \psi_{123} = \psi_{123} \qquad P_{12}\ \psi_{123} = \psi_{213}$$
$$P_{123}\ \psi_{123} = \psi_{231} \qquad P_{12}\,P_{123}\ \psi_{123} = \psi_{132}$$
$$P_{123}^2\ \psi_{123} = \psi_{312} \qquad P_{12}\,P_{123}^2\ \psi_{123} = \psi_{321}$$



Cayley graph

Find combinations transforming under **irreps** of S_3 :

- **Singlet** $\mathcal{S}$ is invariant under any permutation:

$$P_{12}\,\mathcal{S} = \mathcal{S} \qquad \mathcal{S} = \psi_{123} + \psi_{231} + \psi_{312} + \psi_{213} + \psi_{132} + \psi_{321}$$
$$P_{123}\,\mathcal{S} = \mathcal{S}$$

- **Antisinglet** $\mathcal{A}$ is antisymmetric under any transposition i $\leftrightarrow$ j:

$$P_{12}\,\mathcal{A} = -\mathcal{A} \qquad \mathcal{A} = \psi_{123} + \psi_{231} + \psi_{312} - \psi_{213} - \psi_{132} - \psi_{321}$$
$$P_{123}\,\mathcal{A} = \mathcal{A}$$

- **Doublets** $\mathcal{D}_1, \mathcal{D}_2$ form 2-dim subspace:

$$P_{12}\,\mathcal{D}_j = \mathbf{M}_{12}^{\mathrm{T}}\,\mathcal{D}_j \qquad \mathbf{M}_{12} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$P_{123}\,\mathcal{D}_j = \mathbf{M}_{123}^{\mathrm{T}}\,\mathcal{D}_j \qquad \mathbf{M}_{123} = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$$

$$\mathcal{D}_1 = \begin{bmatrix} (\psi_{231} - \psi_{132}) - (\psi_{312} - \psi_{321}) \\ \frac{1}{\sqrt{3}}\left((\psi_{231} + \psi_{132}) + (\psi_{312} + \psi_{321}) - 2(\psi_{123} + \psi_{213})\right) \end{bmatrix}$$

$$\mathcal{D}_2 = \begin{bmatrix} -\frac{1}{\sqrt{3}}\left((\psi_{231} - \psi_{132}) + (\psi_{312} - \psi_{321}) - 2(\psi_{123} - \psi_{213})\right) \\ (\psi_{231} + \psi_{132}) - (\psi_{312} + \psi_{321}) \end{bmatrix}$$

Permutation group S_3

Let's abbreviate $x_1 = {p_1}^2,\ x_2 = {p_2}^2,\ x_3 = {p_3}^2,$ start with $\psi_{123} = x_3$:

$$\begin{aligned} \psi_{123} &= x_3 & \psi_{213} &= x_3 \\ \psi_{231} &= x_1 & \psi_{132} &= x_2 \\ \psi_{312} &= x_2 & \psi_{321} &= x_1 \end{aligned}$$

$\Rightarrow$ only $\boldsymbol{\mathcal{S}}$ and $\boldsymbol{\mathcal{D}}_1$ survive, while $\boldsymbol{\mathcal{A}} = 0$ and $\boldsymbol{\mathcal{D}}_2 = 0$: three variables!

$\Rightarrow$ Singlet: $\mathcal{S}_0 = \frac{x_1+x_2+x_3}{6}$

Doublet: $\mathcal{D} = \begin{bmatrix} a \\ s \end{bmatrix}, \quad a = \sqrt{3}\,\frac{x_1-x_2}{x_1+x_2+x_3}, \quad s = \frac{x_1+x_2-2x_3}{x_1+x_2+x_3}$

$$\boldsymbol{\mathcal{S}} = \psi_{123} + \psi_{231} + \psi_{312} + \psi_{213} + \psi_{132} + \psi_{321}$$

$$\boldsymbol{\mathcal{A}} = \psi_{123} + \psi_{231} + \psi_{312} - \psi_{213} - \psi_{132} - \psi_{321}$$

$$\boldsymbol{\mathcal{D}}_1 = \begin{bmatrix} (\psi_{231} - \psi_{132}) - (\psi_{312} - \psi_{321}) \\ \frac{1}{\sqrt{3}}\left((\psi_{231} + \psi_{132}) + (\psi_{312} + \psi_{321}) - 2(\psi_{123} + \psi_{213})\right) \end{bmatrix}$$

$$\boldsymbol{\mathcal{D}}_2 = \begin{bmatrix} -\frac{1}{\sqrt{3}}\left((\psi_{231} - \psi_{132}) + (\psi_{312} - \psi_{321}) - 2(\psi_{123} - \psi_{213})\right) \\ (\psi_{231} + \psi_{132}) - (\psi_{312} + \psi_{321}) \end{bmatrix}$$

Three-gluon vertex

Let's abbreviate $x_1 = {p_1}^2,\ x_2 = {p_2}^2,\ x_3 = {p_3}^2,$ start with $\psi_{123} = x_3$:

$$\begin{aligned} \psi_{123} &= x_3 & \psi_{213} &= x_3 \\ \psi_{231} &= x_1 & \psi_{132} &= x_2 \\ \psi_{312} &= x_2 & \psi_{321} &= x_1 \end{aligned}$$

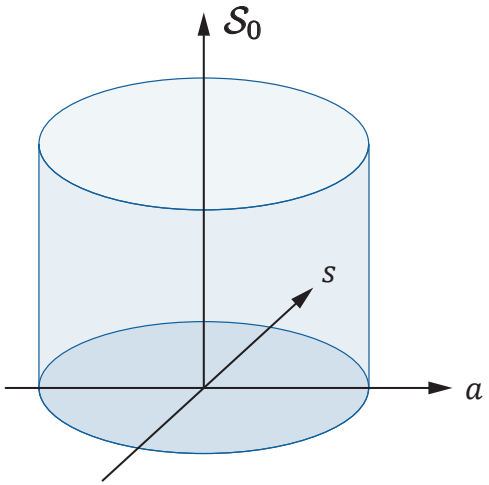
$\Rightarrow$ only $\mathcal{S}$ and $\mathcal{D}_1$ survive, whereas $\mathcal{A} = 0$ and $\mathcal{D}_2 = 0$: three variables!

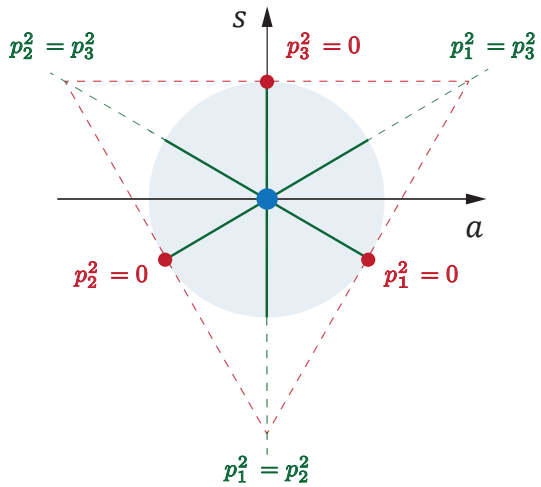
$\Rightarrow$ Singlet: $\mathcal{S}_0 = \frac{x_1+x_2+x_3}{6}$

Doublet: $\mathcal{D} = \begin{bmatrix} a \\ s \end{bmatrix}, \quad a = \sqrt{3}\,\frac{x_1-x_2}{x_1+x_2+x_3}, \quad s = \frac{x_1+x_2-2x_3}{x_1+x_2+x_3}$

For spacelike momenta (like in loop integrations), the kinematic phase space is a **cylinder:**

$$\mathcal{S}_0 > 0$$
$$a^2 + s^2 < 1$$





Three-gluon vertex

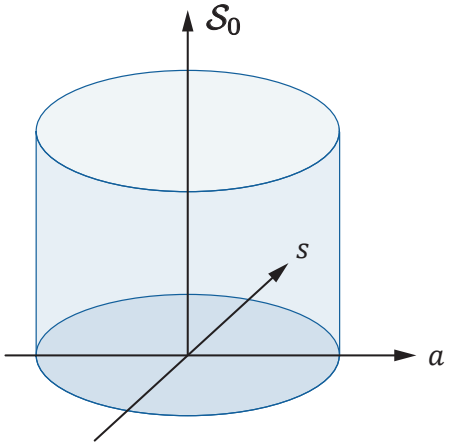
Singlet $\mathcal{S}_0$, Doublet $\mathcal{D} = \begin{bmatrix} a \\ s \end{bmatrix}$, $\quad a = r\,\sin\varphi$, $\quad s = -r\,\cos\varphi$

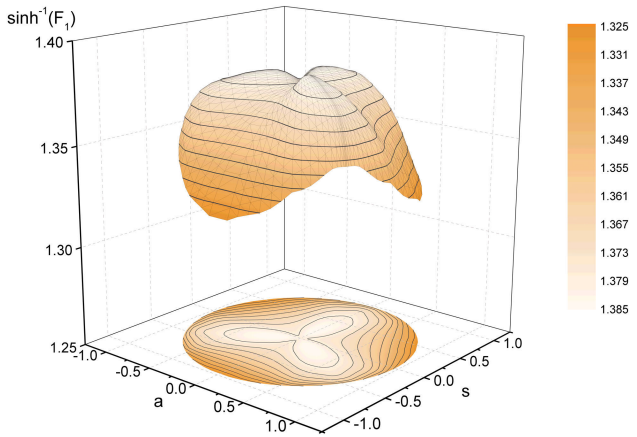
If tensor basis is arranged in antisinglets,
dressing functions must be fully symmetric
⇒ can only depend on **symmetric variables:**

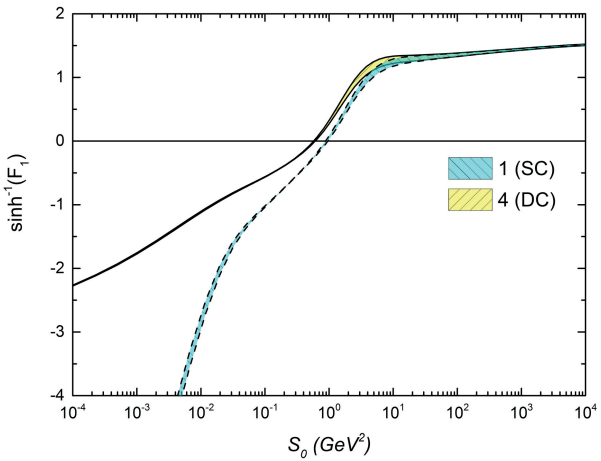
$$\mathcal{S}_0 > 0$$
$$\mathcal{S}_1 = a^2 + s^2 = r^2 < 1$$
$$\mathcal{S}_2 = 3a^2 s - s^3 = r^3 \cos 3\varphi \in [-1, 1]$$

In practice, angular dependence is very weak.
Here: leading dressing function of three-gluon vertex
GE, Williams, Alkofer, Vujinovic, PRD 89 (2014)

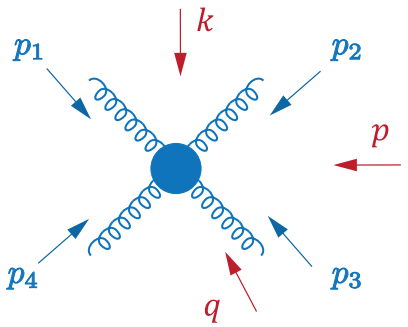
"Planar degeneracy"
Ferreira, Papavassiliou, Particles 6 (2023),
Aguilar, Ferreira, Papavassiliou, Santos, EPJ C 83 (2023)







Four-point functions?



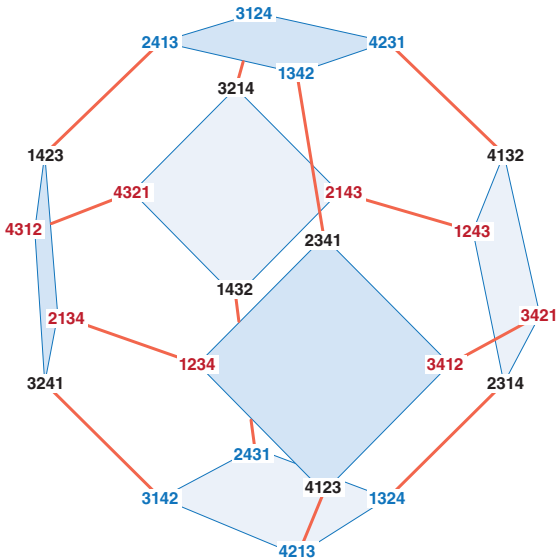
6 Lorentz invariants – which ones?

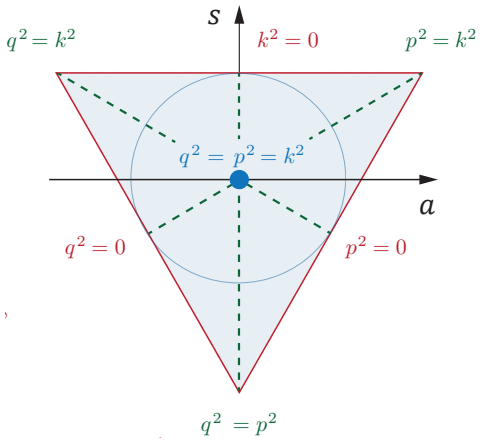
- $p_1^2,\ p_2^2,\ p_3^2,\ p_4^2,\ p_1 \cdot p_2,\ p_3 \cdot p_4$? ... not efficient
- $p^2,\ q^2,\ k^2,\ p \cdot q,\ q \cdot k,\ k \cdot p$? ... better, but still no definite symmetry

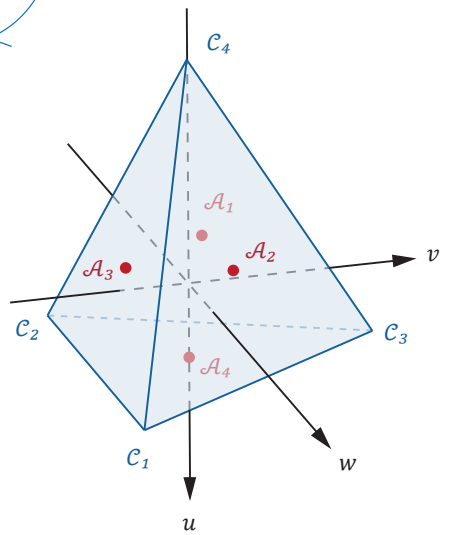
6 variables can be arranged into singlet $\mathcal{S}_0$, doublet $\mathcal{D}$ and triplet $\mathcal{T}$:
GE, Fischer, Heupel, PRD 92 (2015)

If all external legs are onshell, the doublet becomes the **Mandelstam plane** from earlier

Permutation group S_4 has 24 permutations:







Dynamical calculations: Huber, PRD 101 (2020), Aguilar, Ferreira, Papavassiliou, Santos, EPJ C 84 (2024)

Summary

$$G^{\mu\nu\ldots}_{\alpha\beta\ldots}(p_1,\ldots p_n) = \sum_{i=1}^{N} f_i(p_1^2, p_2^2, \ldots)\, \tau_i(p_1,\ldots p_n)^{\mu\nu\ldots}_{\alpha\beta\ldots}$$

Physics

Don't be afraid of Dirac and Lorentz indices!

- Symmetries help
- They make your life easier
- They make your code faster
- They make you smarter
- They are **fun** to play with

1. Introduction
2. Spectrum & symmetries
3. Quark models
4. QFT toolbox
5. Functional methods
6. n-point functions & tensor bases
7. **Multiquark states**
8. Light-front wave functions

Lecture notes "QCD and hadron physics",
https://particle.uni-graz.at/en/quarks-hadrons-and-nuclei

GE, "*Hadron physics with functional methods*",
arXiv: 2503.10397 (for Encyclopedia of Particle Physics)

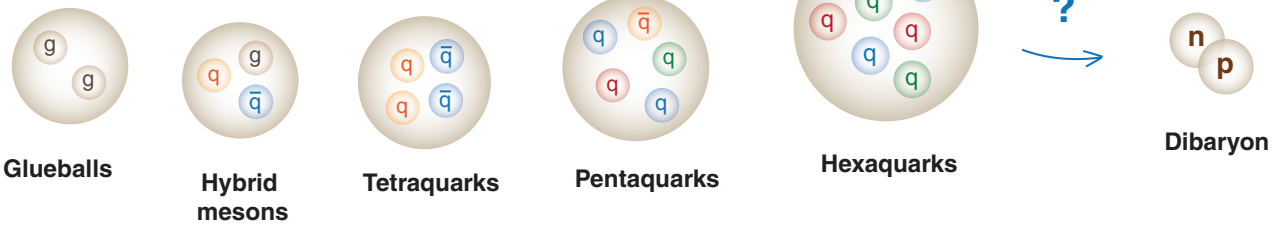
Burkert, GE, Klempt, "*The impact of* γN *and* γ*N *interactions on our understanding of nucleon excitations*", arXiv:2506.16482

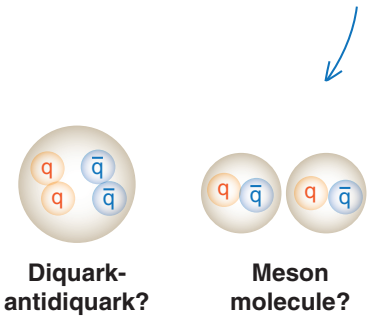
GE, Sanchis-Alepuz, Williams, Alkofer, Fischer,
"*Baryons as relativistic three-quark bound states*",
Prog. Part. Nucl. Phys. 91 (2016), arXiv: 1606.09602

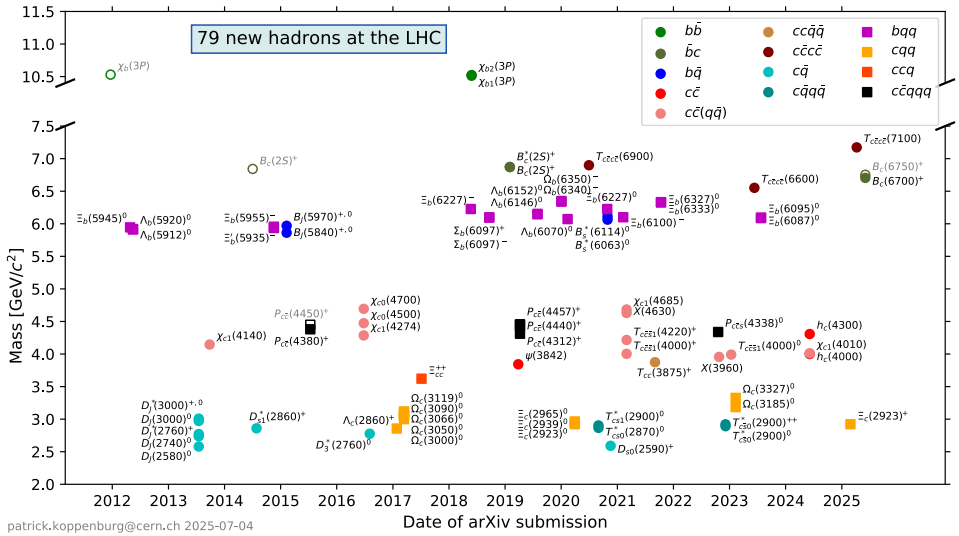
Barabanov, Bedolla, Brooks, Cates, Chen et al.,
"*Diquark correlations in hadron physics: Origin, impact and evidence*", Prog. Part. Nucl. Phys. 116 (2021),
arXiv: 2008.07630

Motivation

- **Understanding exotic hadrons**

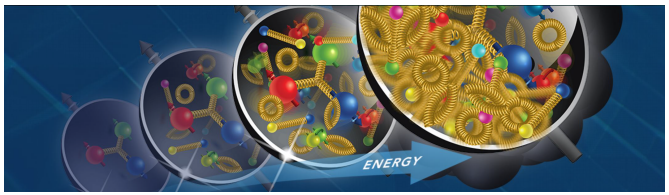




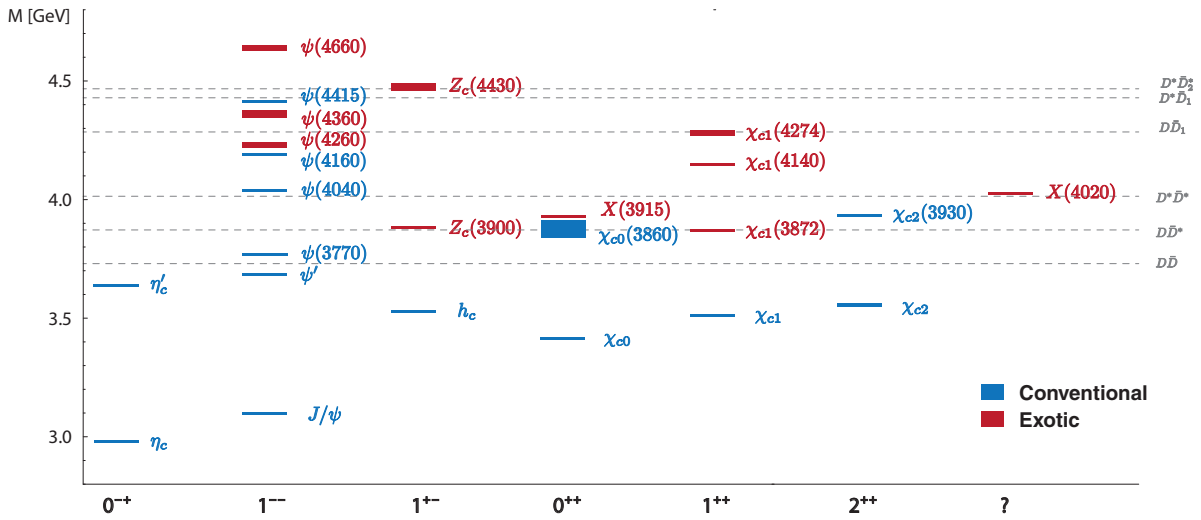


https://www.nikhef.nl/~pkoppenb/particles.html
https://qwg.ph.nat.tum.de/exoticshub

- **Quark-gluon structure of hadrons & nuclei:**
  Hadron tomography at EIC, JLab, AMBER, ...



Exotic mesons



- Several tetraquark candidates in **charmonium spectrum**: X(3872), X(3915), Zc(3900), ...
- Z states cannot be $c\bar{c}$ since they carry charge
- Recent additions: all-charm X(6900), open-charm T_{cc}^{+}, ...
- Oldest tetraquark candidates: **light scalar mesons**

Reviews:

Chen, Chen, Liu, Zhu,
Phys. Rept. 639 (2016), 1601.02092

Lebed, Mitchell, Swanson
PPNP 93 (2017), 1610.04528

Esposito, Pilloni, Polosa,
Phys. Rept 668 (2017), 1611.07920

Guo, Hanhart, Meißner et al.,
Rev. Mod. Phys. 90 (2018), 1705.00141

Ali, Lange, Stone,
PPNP 97 (2017), 1706.00610

Olsen, Skwarnicki, Zieminska,
Rev. Mod. Phys. 90 (2019), 1708.04012

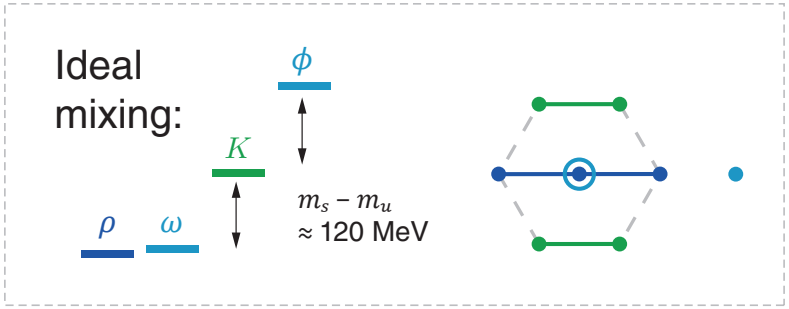
Liu, Chen, Chen, Liu, Zhu,
PPNP 107 (2019), 1903.11976

Brambilla, Eidelman, Hanhart et al.,
Phys. Rept. 873 (2020)

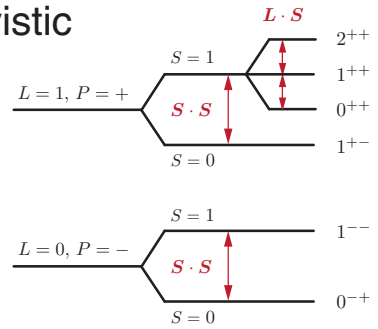
...

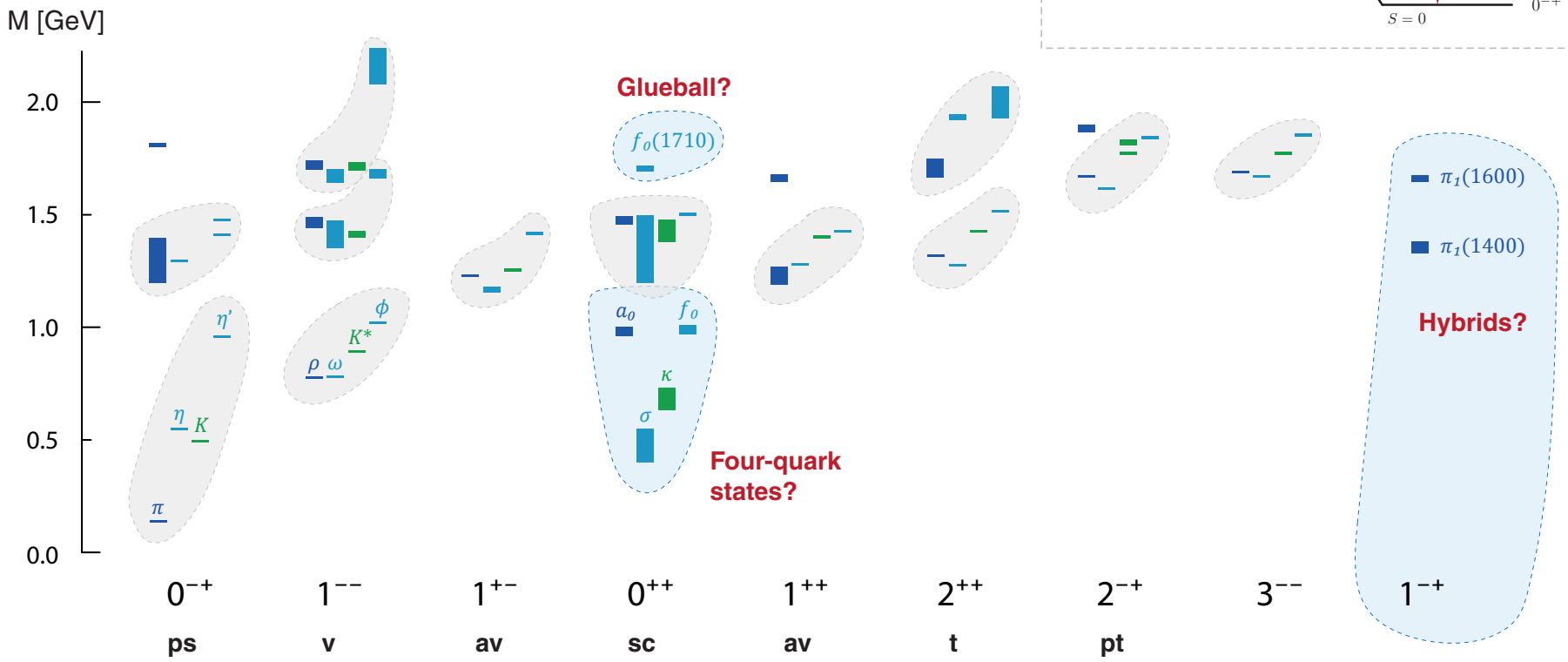
Light exotic mesons

Light meson spectrum
(PDG 2020)



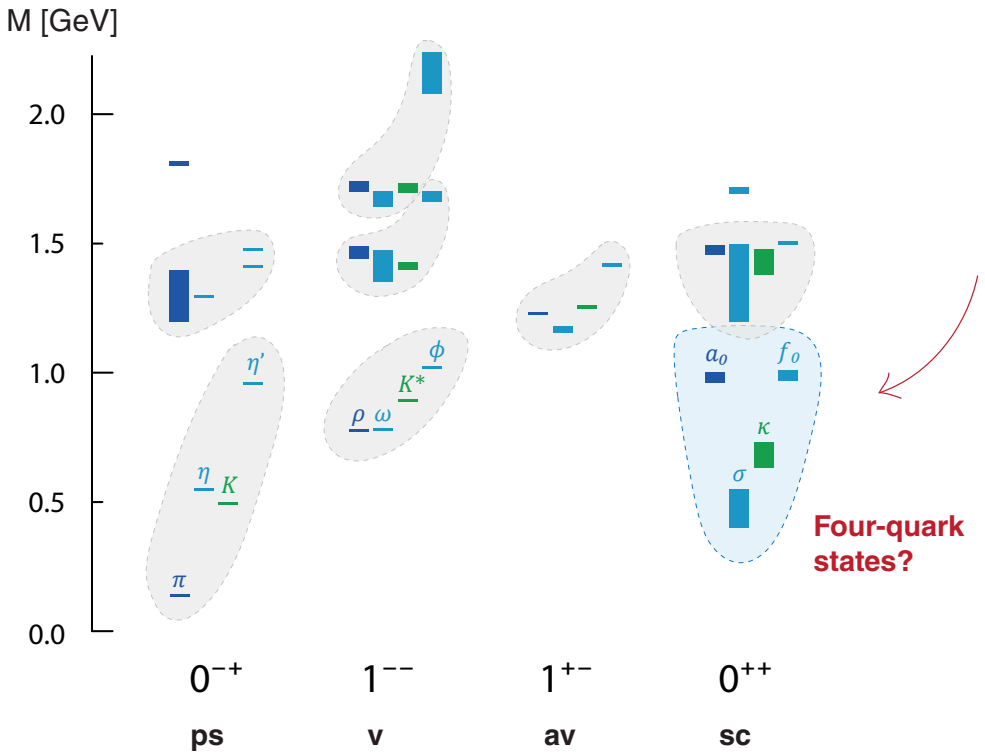
Non-relativistic level ordering



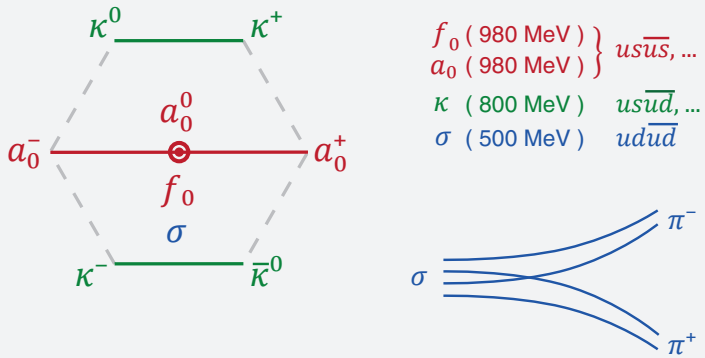


Light exotic mesons

Light meson spectrum
(PDG 2020)



- **Diquark-antidiquark**?
  Explains mass ordering & decay widths
  Jaffe 1977, Close, Tornqvist 2002,
  Maiani, Polosa, Riquer 2004

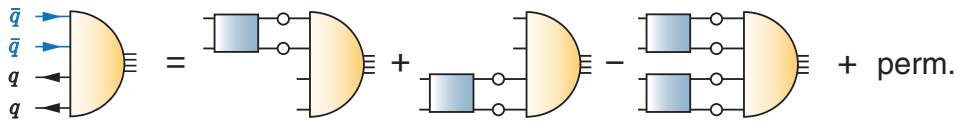


- **Meson molecules?**
  Weinstein, Isgur 1982, 1990; Close, Isgur, Kumano 1993

- **Non-$q\bar{q}$ nature** supported by various approaches
  Pelaez, Phys. Rept. 658 (2016)

Four-quark states

- Light scalar mesons (σ, κ, a_0, f_0) as **four-quark states:**
  GE, Fischer, Heupel, PLB 753 (2016)



$$\Gamma(p,q,k,P) = \sum_i f_i\left(p^2,q^2,k^2,\{\omega_j\},\{\eta_j\}\right)\,\tau_i(p,q,k,P) \;\otimes\; \text{Color} \;\otimes\; \text{Flavor}$$

9 Lorentz invariants:

$p^2,\quad q^2,\quad k^2,\quad P^2 = -M^2$

$\omega_1 = q\cdot k \qquad \eta_1 = p\cdot P$
$\omega_2 = p\cdot k \qquad \eta_2 = q\cdot P$
$\omega_3 = p\cdot q \qquad \eta_3 = k\cdot P$

256 Dirac-Lorentz tensors

2 Color tensors:

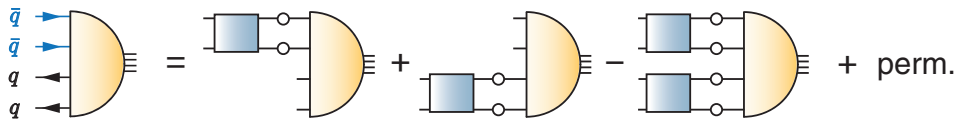
$3\otimes\bar{3},\quad 6\otimes\bar{6}$ or
$1\otimes 1,\quad 8\otimes 8$
(Fierz-equivalent)

$K\,\psi_i = \lambda_i\,\psi_i$

	dim K	memory
Mesons	10^3	20 MB
Baryons	10^8	10^7 GB
Tetraquarks	10^{13}	10^{18} GB

Four-quark states

- Light scalar mesons (σ, κ, a_0, f_0) as **four-quark states:**
  GE, Fischer, Heupel, PLB 753 (2016)



$$\Gamma(p,q,k,P) = \sum_i f_i\left(p^2, q^2, k^2, \{\omega_j\}, \{\eta_j\}\right) \tau_i(p,q,k,P) \otimes \text{Color} \otimes \text{Flavor}$$

9 Lorentz invariants:

$p^2, \quad q^2, \quad k^2, \quad P^2 = -M^2$

$\omega_1 = q \cdot k \qquad \eta_1 = p \cdot P$
$\omega_2 = p \cdot k \qquad \eta_2 = q \cdot P$
$\omega_3 = p \cdot q \qquad \eta_3 = k \cdot P$

256 Dirac-Lorentz tensors

2 Color tensors:

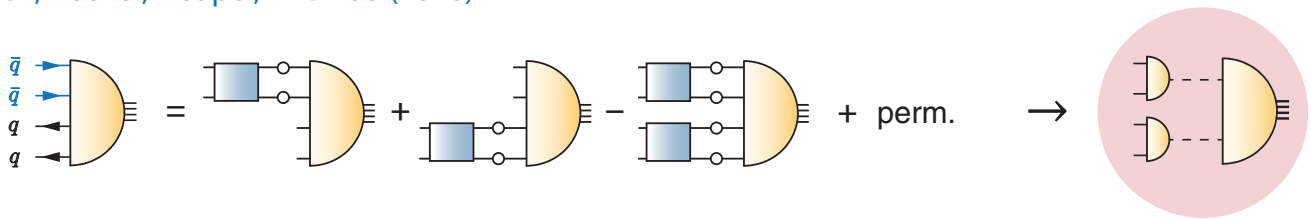
$3 \otimes \bar{3}, \quad 6 \otimes \bar{6}$ or
$1 \otimes 1, \quad 8 \otimes 8$
(Fierz-equivalent)

- Group momentum variables into multiplets of **permutation group S4:** can switch off groups of variables without destroying symmetries
  GE, Fischer, Heupel, PRD 92 (2015)

$f_i\left(\mathcal{S}_0, \bigtriangledown, \diamondsuit, \bigcirc\right)$

Four-quark states

- Light scalar mesons (σ, κ, a_0, f_0) as **four-quark states:**
  GE, Fischer, Heupel, PLB 753 (2016)



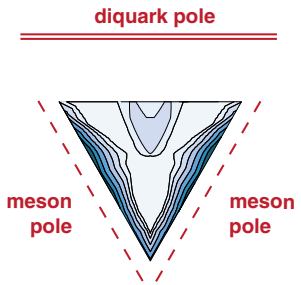
- BSE dynamically generates **meson poles** in BS amplitude:

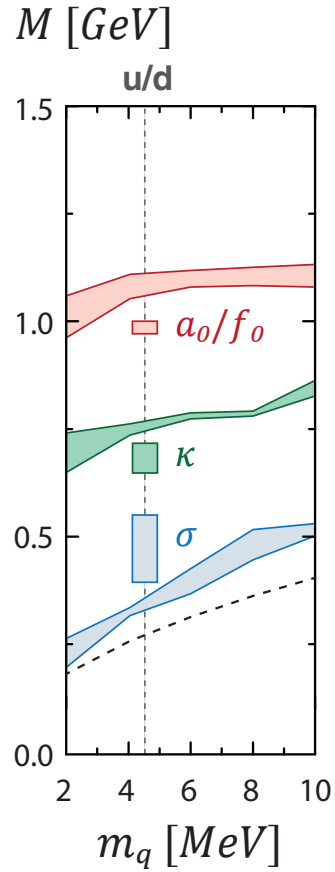
$f_i(\mathcal{S}_0, \triangledown, \triangle, \bigcirc) \rightarrow$ 1500 MeV

$f_i(\mathcal{S}_0, \triangledown, \triangle, \bigcirc) \rightarrow$ 1500 MeV

$f_i(\mathcal{S}_0, \triangledown, \triangle, \bigcirc) \rightarrow$ 1200 MeV

$f_i(\mathcal{S}_0, \triangledown, \triangle, \bigcirc) \rightarrow$ **350 MeV !**



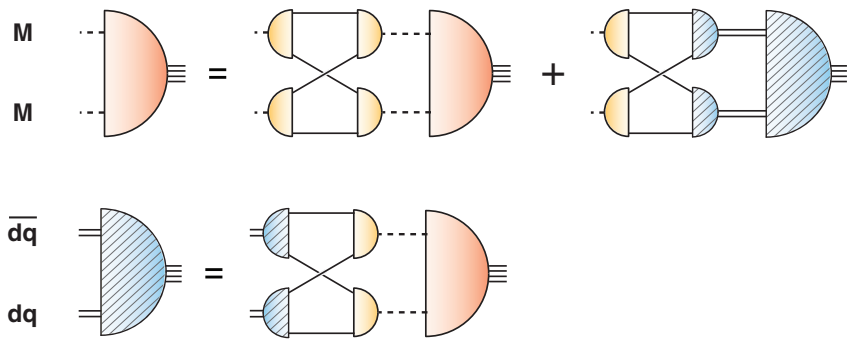


- "Light scalar mesons" look like **meson molecules,**
  diquark-antidiquark components almost negligible.
  Lightness is inherited from pseudoscalar Goldstone bosons!

Four-quark states

Two-body formulation: **meson-meson / diquark-antidiquark,**
follows from four-quark eq. (analogue of quark-diquark for baryons)

Heupel, GE, Fischer, PLB 718 (2012)

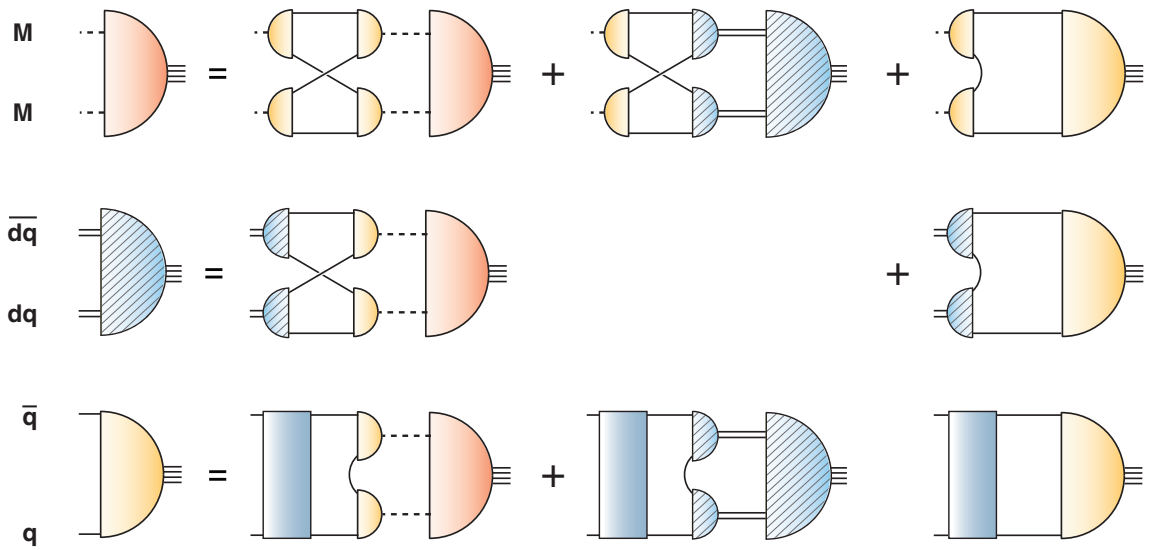


- Interaction by **quark exchange**
- System 'wants' to be **meson-meson**-like (no diagonal dq-$\overline{dq}$ term)
- Similar results as in 4-quark approach: m_σ ~ 400 MeV, etc.

Four-quark states

Two-body formulation: **meson-meson / diquark-antidiquark,**
follows from four-quark eq. (analogue of quark-diquark for baryons)

Heupel, GE, Fischer, PLB 718 (2012)



Include **mixing with $q\bar{q}$:**
$\pi\pi$ still dominant

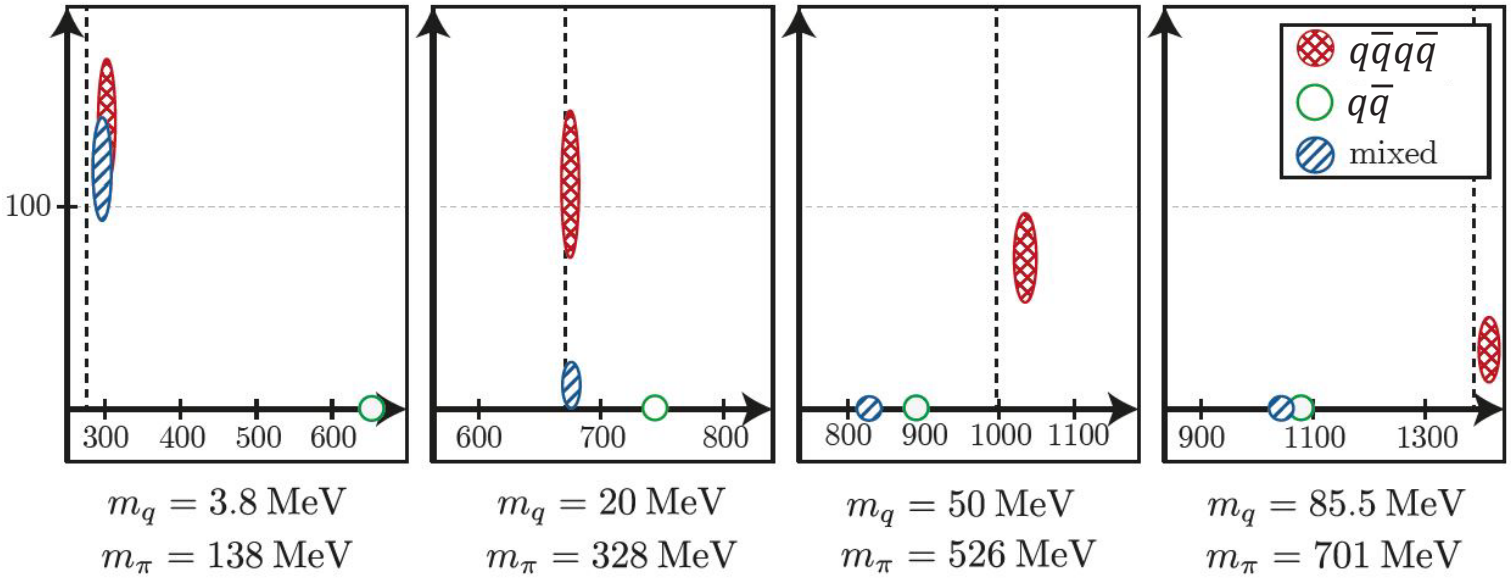
Santowsky, GE, Fischer, Wallbott, Williams, PRD 102 (2020)

[MeV]	ground state mass	first excitation
$\pi\pi$	416 ± 26	970 ± 130
$\pi\pi + 0^+0^+$	416 ± 26	970 ± 130
$q\bar{q}$	667 ± 2	1036 ± 8
$\pi\pi + q\bar{q}$	472 ± 22	1080 ± 280
$\pi\pi + 0^+0^+ + q\bar{q}$	456 ± 24	1110 ± 110

Four-quark states

Four-quark vs. $q\bar{q}$ dominance

Santowsky, Fischer, PRD 105 (2022)



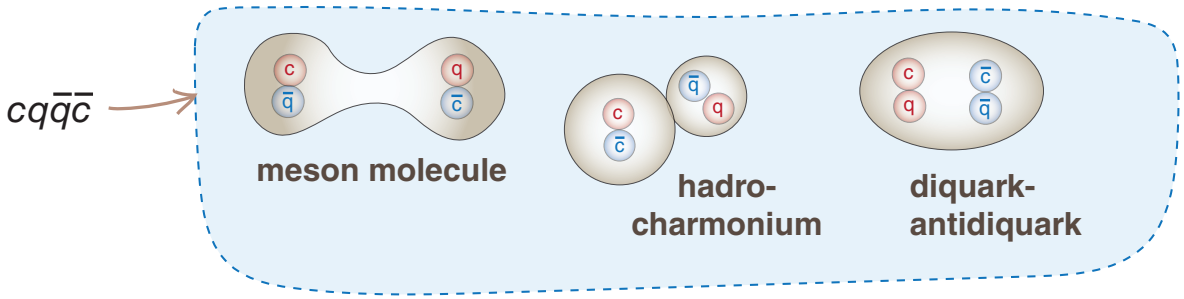
Light quarks:
$\pi\pi$ dominant

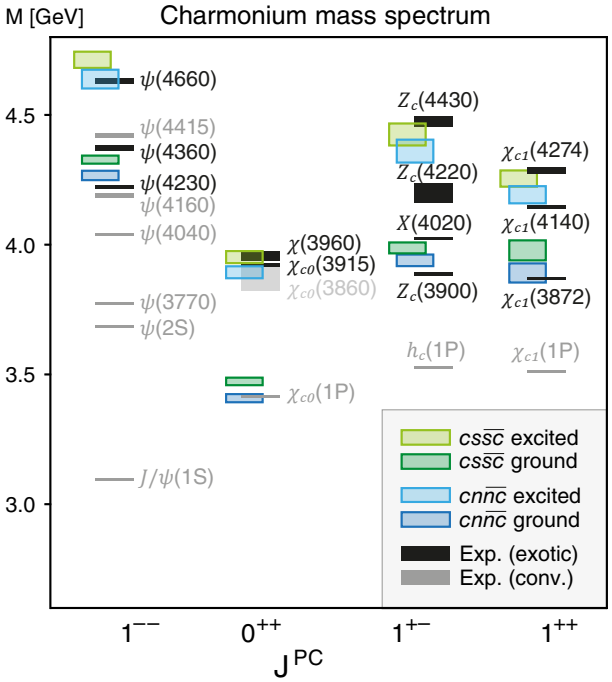
Strange quarks:
$q\bar{q}$ dominant

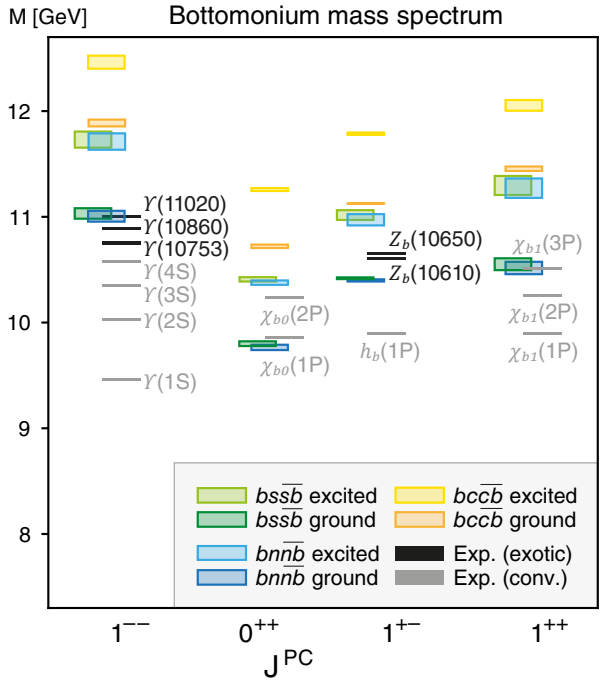
Four-quark states

Hidden charm, hidden bottom

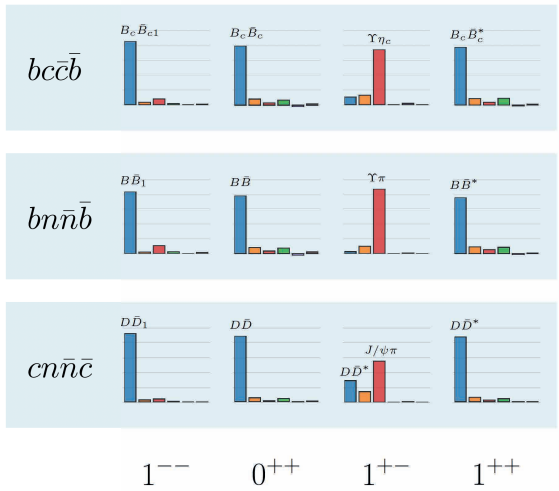
Hoffer, GE, Fischer, PRD 109 (2024)







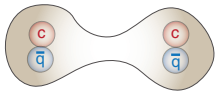
Wave-function components:



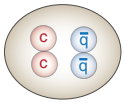
Four-quark states

Open-flavor states

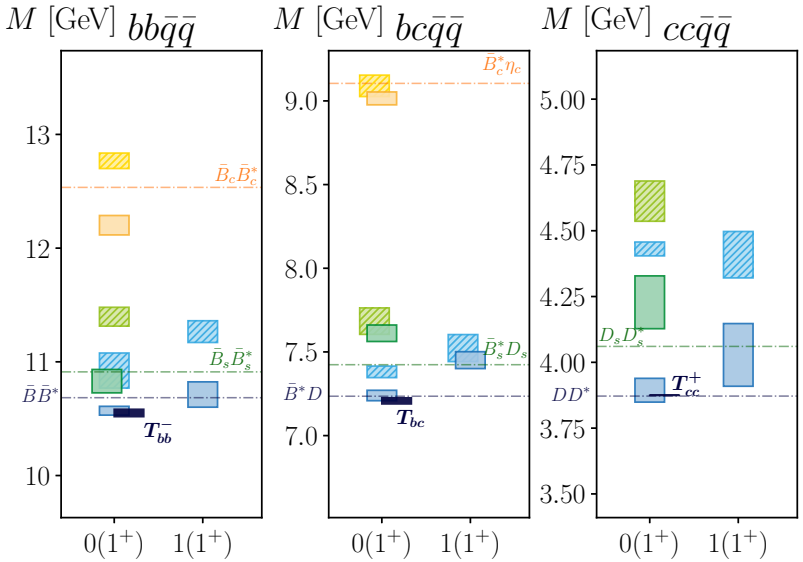
Hoffer, GE, Fischer,
PRD 111 (2025)

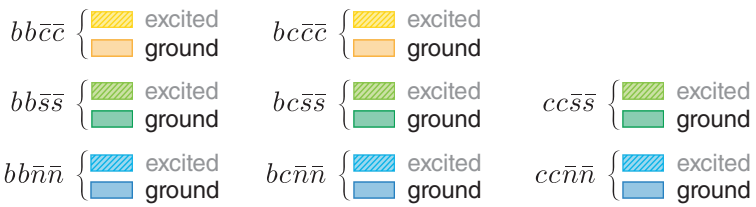


meson molecule

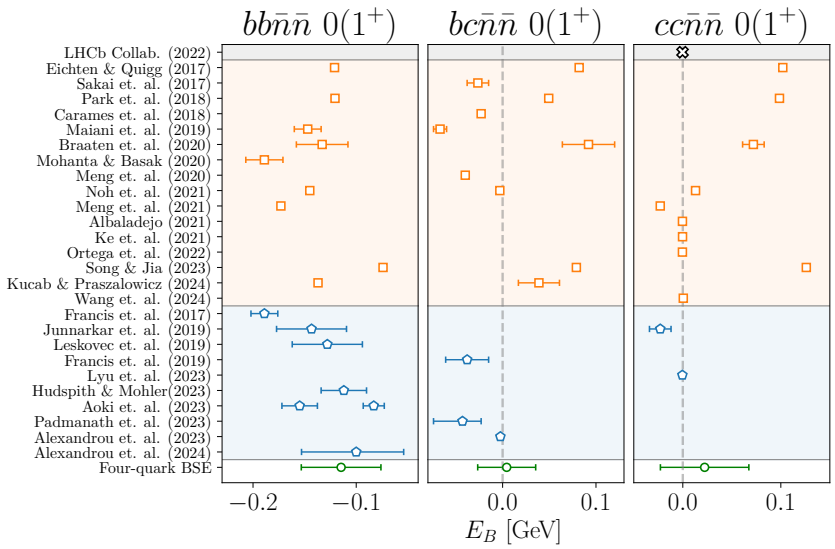


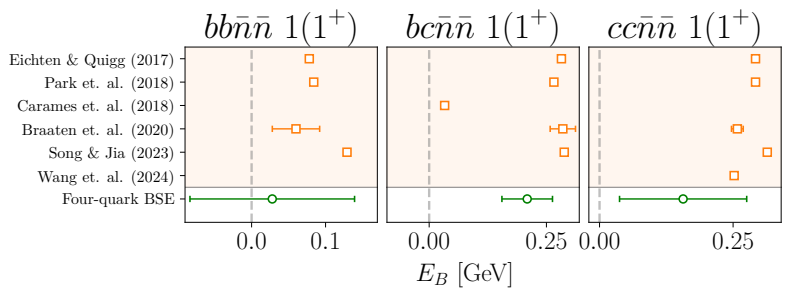
diquark-antidiquark





Binding energies (ground states):

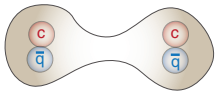




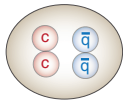
Four-quark states

Open-flavor states

Hoffer, GE, Fischer,
PRD 111 (2025)

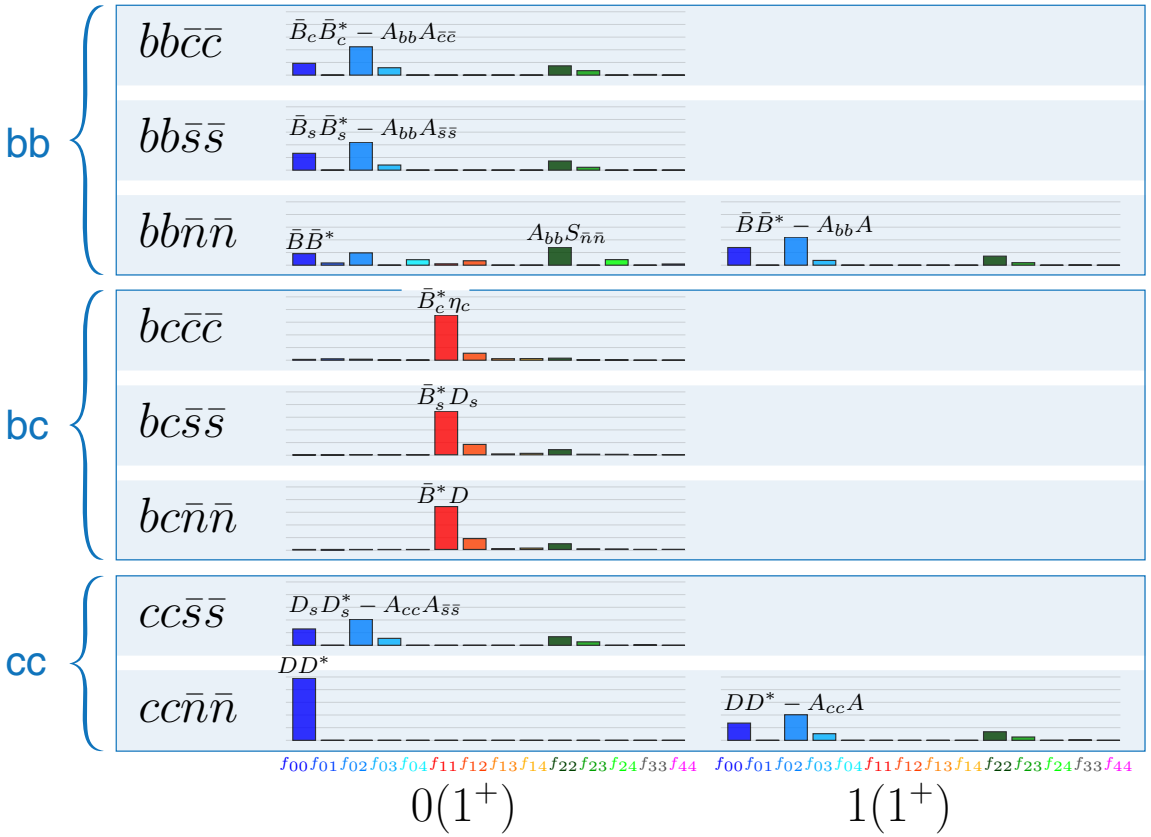


meson molecule



diquark-antidiquark

Wave-function components:



$I(J^P)$	Physical components					
	$\mathbf{1}\otimes\mathbf{1}$		$\bar{\mathbf{3}}\otimes\mathbf{3}$	$\mathbf{8}\otimes\mathbf{8}$		$\bar{\mathbf{6}}\otimes\mathbf{6}$
	f_0	f_1	f_2	f_3	f_4	f_5
$0(1^+)$ $bb\bar{n}\bar{n}$	BB^*	B^*B^*	$A_{bb}S$	BB^*	B^*B^*	$S_{bb}A$
$bc\bar{n}\bar{n}$	BD^*	B^*D	$A_{bc}S$	BD^*	B^*D	$S_{bc}A$
$cc\bar{n}\bar{n}$	DD^*	D^*D^*	$A_{cc}S$	DD^*	D^*D^*	$S_{cc}A$
$bb\bar{s}\bar{s}$	$B_sB_s^*$	–	$A_{bb}A_{ss}$	$B_sB_s^*$	–	–
$bc\bar{s}\bar{s}$	$B_sD_s^*$	$B_s^*D_s$	$S_{bc}A_{ss}$	$B_sD_s^*$	$B_s^*D_s$	$A_{bc}S_{ss}$
$cc\bar{s}\bar{s}$	$D_sD_s^*$	–	$A_{cc}A_{ss}$	$D_sD_s^*$	–	–
$1(1^+)$ $bb\bar{q}\bar{q}$	BB^*	–	$A_{bb}A$	BB^*	–	–
$cc\bar{q}\bar{q}$	DD^*	–	$A_{cc}A$	DD^*	–	–

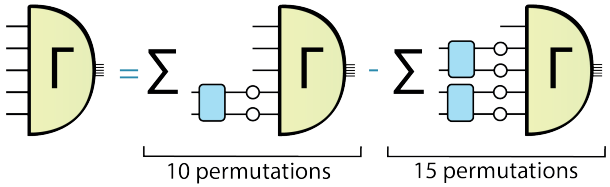
→ T_{bb} is complicated mixture of molecule and diquark-antidiquark

Can **quantify** meson-meson and diquark-antidiquark contributions

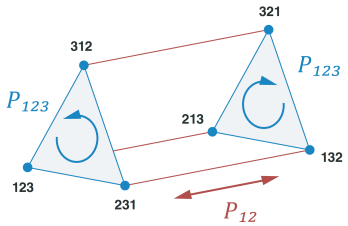
→ T_{cc} sits on top of threshold, pure molecule

Five-body systems

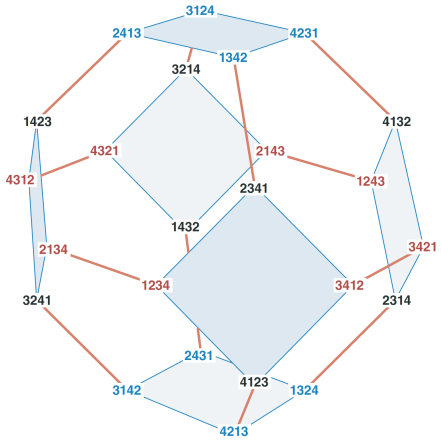
- **Five-body equation**
  GE, Torres, PRD 111 (2025),
  GE, Peña, Torres, PLB 866 (2025)



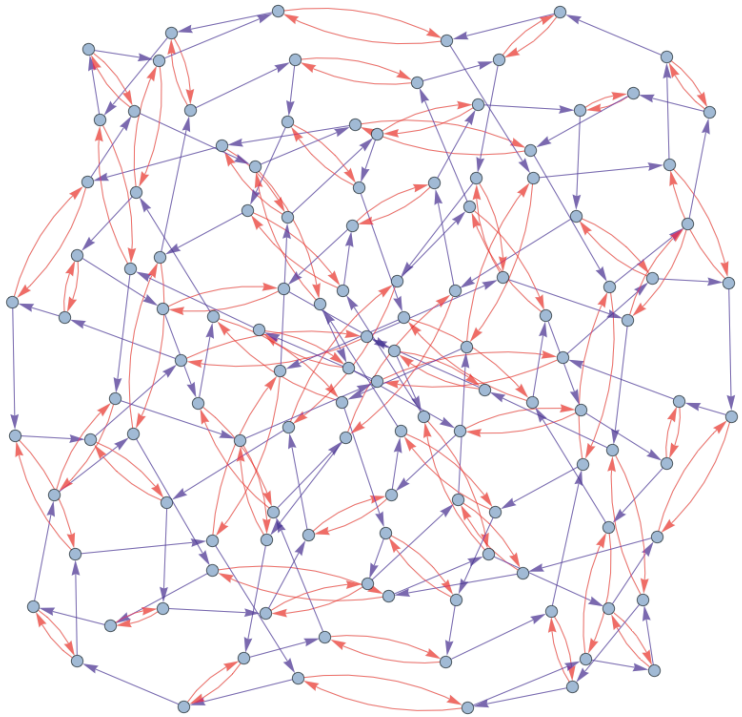
- Multiplets of permutation group S_5?



Cayley graph of S_3



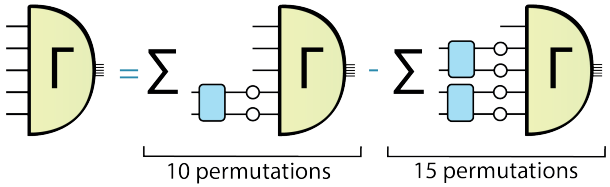
Cayley graph of S_4



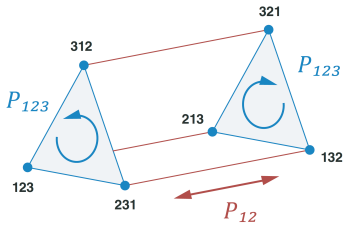
Cayley graph of S_5

Five-body systems

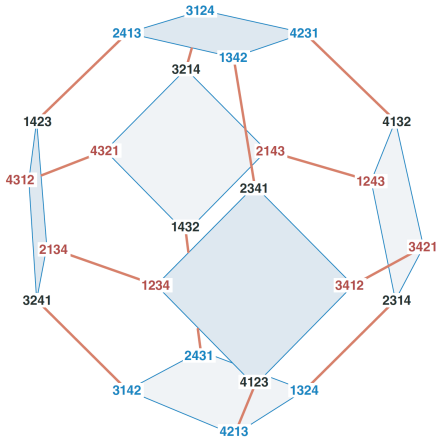
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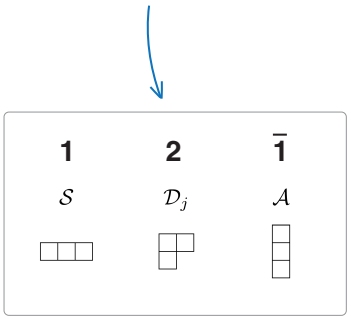
- Multiplets of permutation group S_5?

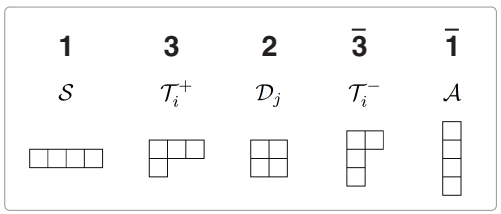


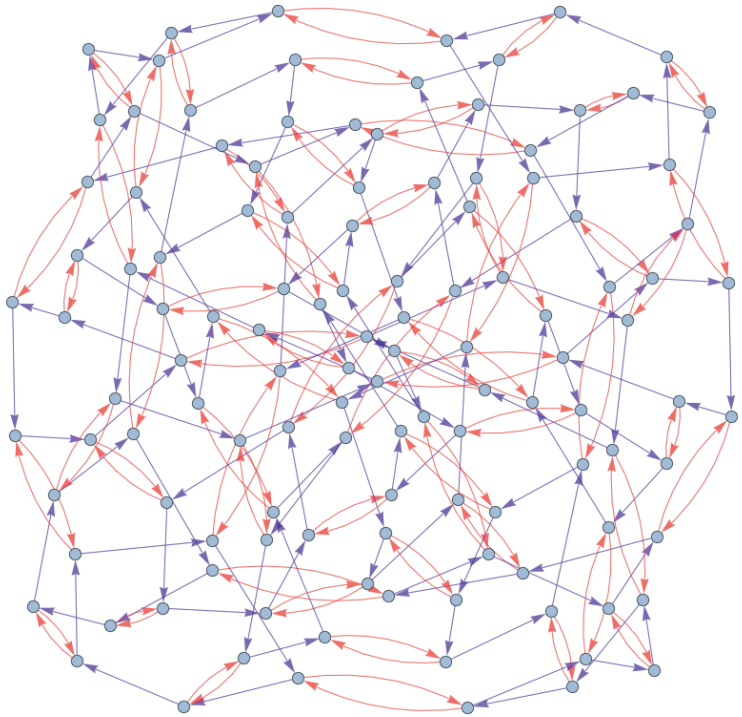
Cayley graph of S_3



Cayley graph of S_4



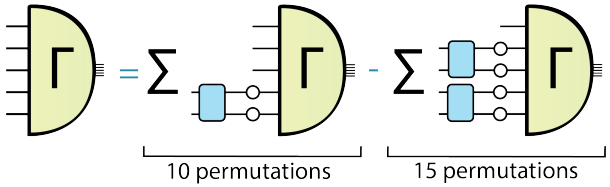




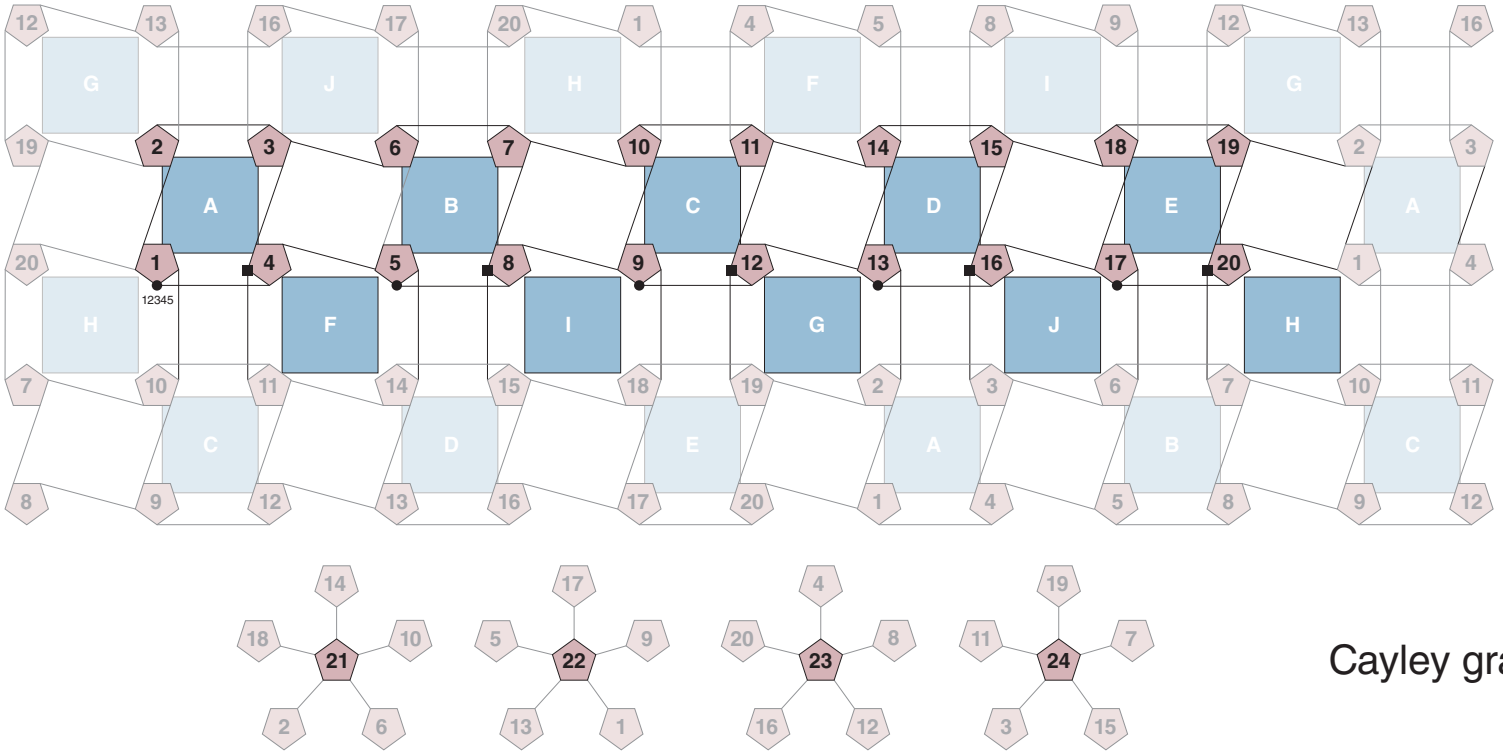
Cayley graph of S_5

Five-body systems

- **Five-body equation**
  GE, Torres, PRD 111 (2025),
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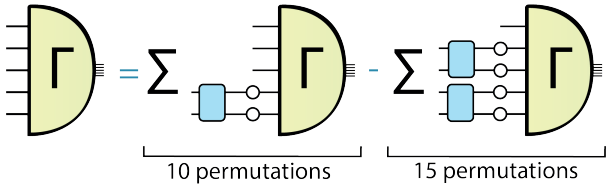
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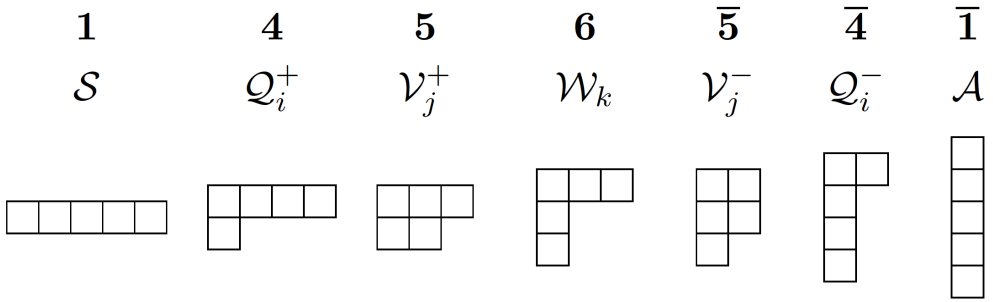
Cayley graph of S_5

Five-body systems

- **Five-body equation**
  GE, Torres, PRD 111 (2025),
  GE, Peña, Torres, PLB 866 (2025)



- Multiplets of permutation group S_5:



- E.g., 32 linearly independent color structures for **five-gluon vertex:**

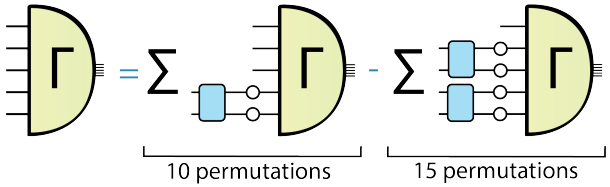
$$\delta_{ab}\, d_{cde} \;\to\; \mathbf{1},\ \mathbf{4},\ \mathbf{5}$$
$$\delta_{ab}\, f_{cde} \;\to\; \overline{\mathbf{4}},\ \mathbf{6}$$
$$f_{abr}\, f_{cds}\, f_{ers} \;\to\; \mathbf{6}$$
$$f_{abr}\, f_{cds}\, d_{ers} \;\to\; \overline{\mathbf{5}},\ \overline{\mathbf{1}}$$

$$\mathcal{Q}^- = \begin{bmatrix} \psi_1 - \psi_2 \\ \frac{1}{\sqrt{3}}(\psi_1 + \psi_2 - 2\psi_3) \\ \frac{1}{\sqrt{6}}(\psi_1 + \psi_2 + \psi_3 - 3\psi_4) \\ \frac{1}{\sqrt{10}}(\psi_1 + x_2 + \psi_3 + \psi_4 - 4\psi_5) \end{bmatrix},$$

$$\mathcal{W} = \begin{bmatrix} 2\left(f_{45} - \frac{1}{5}(\psi_4 - \psi_5)\right) \\ \frac{1}{\sqrt{2}}\left(3f_{35} - f_{45} + \frac{1}{5}(3\psi_3 + \psi_4 - 4\psi_5)\right) \\ \sqrt{\frac{3}{10}}(4f_{34} - f_{35} - f_{45} - \psi_3 + \psi_4) \\ \sqrt{\frac{3}{2}}\left(f_{15} + f_{25} + \frac{1}{5}(\psi_1 - \psi_2)\right) \\ \frac{1}{\sqrt{10}}(-2f_{12} + f_{13} + 3f_{14} + f_{23} + 3f_{24}) \\ \frac{2}{\sqrt{5}}(f_{12} + f_{13} + f_{23}) \end{bmatrix}$$

Five-body systems

- **Five-body equation**
  GE, Torres, PRD 111 (2025),
  GE, Peña, Torres, PLB 866 (2025)



$$\Gamma(p,q,k,l,P) = \sum_i f_i(p^2,q^2,k^2,l^2,P^2,\{\omega_i\},\{\eta_i\})\,\tau_i(p,q,k,l,P)$$

15 Lorentz invariants,
group into multiplets of S_5:
1 + 1 + 4 + 4 + 5

$$\begin{aligned}
\omega_1 &= q\cdot p\,, & & \\
\omega_2 &= q\cdot k\,, & \eta_1 &= q\cdot P\,,\\
\omega_3 &= q\cdot l\,, & \eta_2 &= p\cdot P\,,\\
\omega_4 &= p\cdot k\,, & \eta_3 &= k\cdot P\,,\\
\omega_5 &= p\cdot l\,, & \eta_4 &= l\cdot P\,.\\
\omega_6 &= k\cdot l\,, &&
\end{aligned}$$

+ dimensional constraint: only 14 are independent

$$P=\begin{bmatrix}0\\0\\0\\\bullet\end{bmatrix},\ q=\begin{bmatrix}0\\0\\\bullet\\\bullet\end{bmatrix},\ p=\begin{bmatrix}0\\\bullet\\\bullet\\\bullet\end{bmatrix},\ k=\begin{bmatrix}\bullet\\\bullet\\\bullet\\\bullet\end{bmatrix},\ l=\begin{bmatrix}\bullet\\\bullet\\\bullet\\\bullet\end{bmatrix}$$

unfortunately, complicated constraint equation...

$$\begin{aligned}
&\left[(\omega_{12}\,\omega_{34}+\omega_{13}\,\omega_{24}+\omega_{14}\,\omega_{23})^2-q_1^2\,q_2^2\,q_3^2\,q_4^2\right]P^2\\
&+\sum_i \eta_i^2\left[q_j^2\,q_k^2\,q_l^2-q_j^2\,\omega_{kl}^2-q_k^2\,\omega_{jl}^2-q_l^2\,\omega_{jk}^2\right]\\
&+2\sum_i\left(\eta_i^2-q_i^2\,P^2\right)\omega_{jk}\,\omega_{jl}\,\omega_{kl}\\
&+\sum_{i<j}\eta_i\,\eta_j\left[2q_k^2\left(\omega_{il}\,\omega_{jl}-q_l^2\,\omega_{ij}\right)+q_l^2\,\omega_{ik}\,\omega_{jk}\right.\\
&\qquad\left.+\omega_{kl}\left(\omega_{ij}\,\omega_{kl}-\omega_{ik}\,\omega_{jl}-\omega_{il}\,\omega_{jk}\right)\right]\\
&+P^2\sum_{i<j}\omega_{ij}^2\,\left(q_k^2\,q_l^2-\omega_{kl}^2\right)=0\,,
\end{aligned}$$

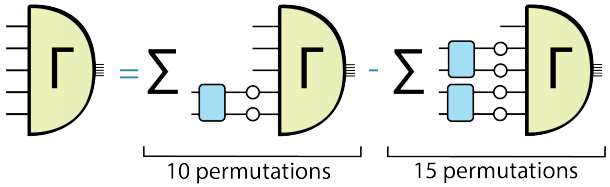
- can be generalized to arbitrary n-body equations:

n	$\mathcal{S}_0$	P^2	η_i	$p_1^2\ldots p_n^2$	$\mathcal{M}$	Total	Indep.
2	1	1	1	–	–	3	3
3	1	1	2	2	–	6	6
4	1	1	3	3	2	10	10
5	1	1	4	4	5	15	14
6	1	1	5	5	9	21	18
n	1	1	$n-1$	$n-1$	$n(n-3)/2$	$n(n+1)/2$	$4n-6$

two-/three-body poles distributed over **4 + 5**

Five-body systems

- **Five-body equation**
  GE, Torres, PRD 111 (2025),
  GE, Peña, Torres, PLB 866 (2025)

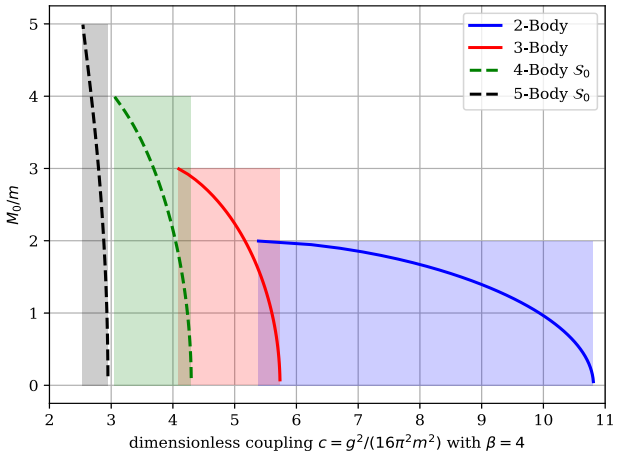


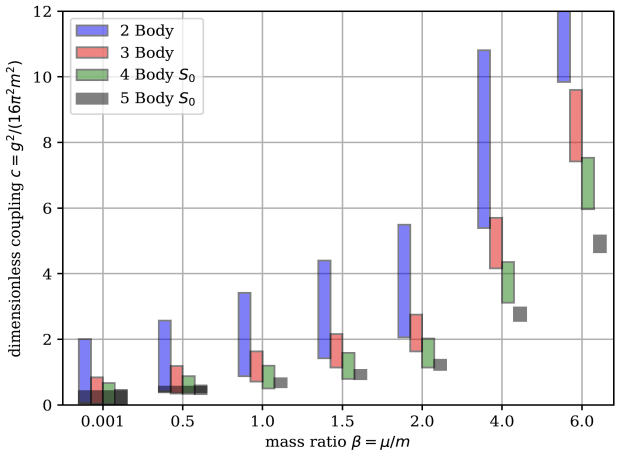
- Approximation: absorb **two-/three-body poles** explicitly in wave function

$$\Gamma(q,p,k,l,P) \approx f(S_0) \sum_{aa'} \mathcal{P}_{aa'} \,, \qquad \mathcal{P}_{(12)(345)} = \frac{1}{(p_1+p_2)^2 + M_M^2} \frac{1}{(p_3+p_4+p_5)^2 + M_B^2}$$

⇒ needs solutions of two-/three-body equations

- Scalar (massive Wick-Cutkosky) model:
  Parameters are coupling strength c, exchange particle mass β

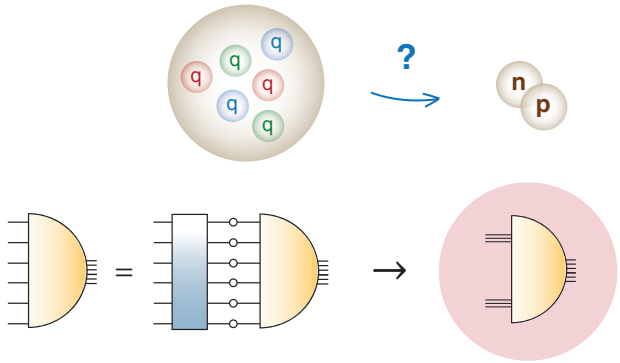




Regions in c, β where system is **Borromean:** 5-body eq. has solution but 2- and 3-body eqs. do not

Six-body systems

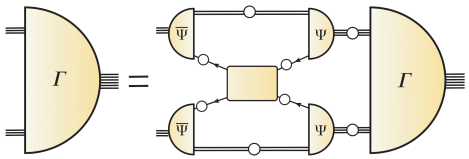
- 6-body equations: Transition from quarks & gluons to **light nuclei**
  GE, Hoffer, in preparation



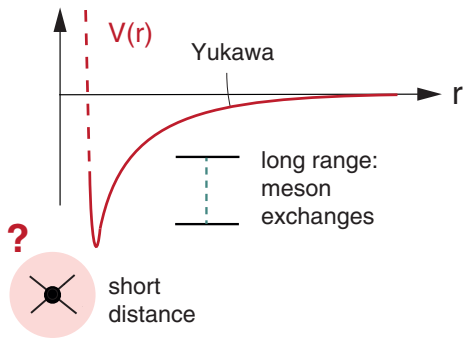
- Wave function depends on 21 variables (**1** + **1** + **5** + **5** + **9**), 3 constraint equations

n	$\mathcal{S}_0$	P^2	η_i	$p_1^2 \ldots p_n^2$	$\mathcal{M}$	Total	Indep.
2	1	1	1	–	–	3	3
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n	1	1	$n-1$	$n-1$	$n(n-3)/2$	$n(n+1)/2$	$4n-6$

- Relativistic structure of the **deuteron**
  Arriaga, GE, Nunes, Peña, Stadler, in preparation



- Microscopic origin of **short-range nuclear force?**
  GE, Mirtl, in preparation



1. Introduction
2. Spectrum & symmetries
3. Quark models
4. QFT toolbox
5. Functional methods
6. n-point functions & tensor bases
7. Multiquark states
8. **Light-front wave functions**

Lecture notes "QCD and hadron physics",
https://particle.uni-graz.at/en/quarks-hadrons-and-nuclei

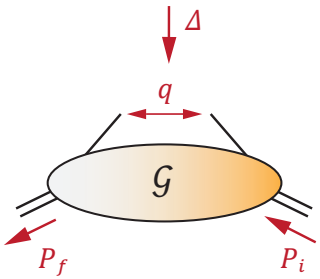
GE, "*Hadron physics with functional methods*",
arXiv: 2503.10397 (for Encyclopedia of Particle Physics)

Burkert, GE, Klempt, "*The impact of* γN *and* γ*N *interactions on our understanding of nucleon excitations*", arXiv:2506.16482

GE, Sanchis-Alepuz, Williams, Alkofer, Fischer,
"*Baryons as relativistic three-quark bound states*",
Prog. Part. Nucl. Phys. 91 (2016), arXiv: 1606.09602

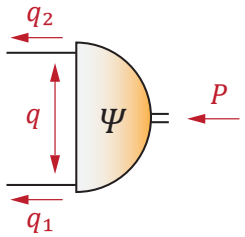
Barabanov, Bedolla, Brooks, Cates, Chen et al.,
"*Diquark correlations in hadron physics: Origin, impact and evidence*", Prog. Part. Nucl. Phys. 116 (2021),
arXiv: 2008.07630

Parton distributions



Hadron-to-hadron correlator

$$\mathcal{G}(z,P,\Delta) = \langle P_f | \,\mathsf{T}\, \Phi(z)\, \mathcal{O}\, \Phi(0)\, | P_i \rangle$$



Bethe-Salpeter WF:
vacuum-to-hadron correlator

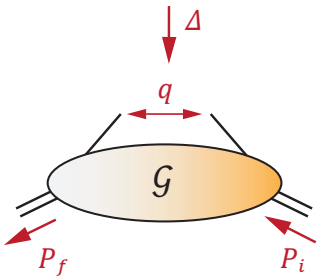
$$\Psi(z,P) = \langle 0 | \,\mathsf{T}\, \Phi(z)\, \Phi(0) | P \rangle$$

	$\mathcal{G}(q,P,\Delta=0)$	$\mathcal{G}(q,P,\Delta)$	$\Psi(q,P)$
$\int dq^-$	TMD	GTMD	LFWF
$\int d^2\boldsymbol{q}_\perp \int dq^-$	PDF	GPD	PDA

Diehl, Phys. Rept. 388 (2003)
Belitsky, Radyushkin,
Phys. Rept. 418 (2005)
Lorcé, Pasquini, Vanderhaeghen,
JHEP 05 (2011)
...

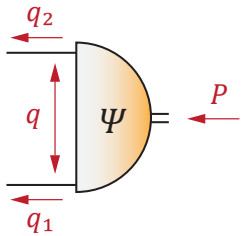
Light-front wave functions:
coefficients of Fock expansion
in light-front quantization
Brodsky, Pauli, Pinsky, Phys. Rept. 301 (1998)

Parton distributions



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Progress in **lattice QCD:**
Quasi-PDFs, pseudo-PDFs, ...

Ji, PRL 110 (2013),
Radyushkin, PLB 767 (2017),
Lin et al., PPNP 100 (2018)
Constantinou et al., PPNP 121 (2021)

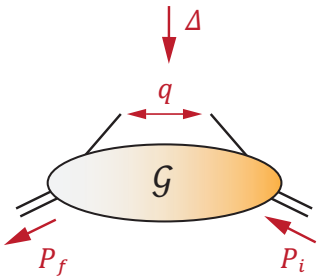
Nakanishi representation

Nakanishi 1963, 1969, 1988
Kusaka, Williams, PRD 51 (1995)
Sauli, Adam, PRD 67 (2003)
Karmanov, Carbonell, EPJ A 27 (2006)
Frederico, Salmè, Viviani, PRD 85 (2012)

Many continuum studies of
LFWFs, PDFs, GPDs, TMDs, ...

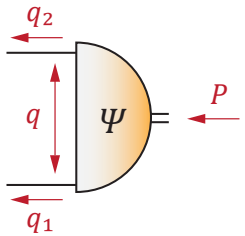
Tiburzi, Miller, PRD 65 (2002)
Nguyen, Bashir, Roberts, Tandy, PRC 83 (2011)
Chang et al., PRL 110 (2013)
Frederico, Salmè, Viviani, PRD 89 (2014)
Mezrag et al., PLB 741 (2015)
de Paula et al., PRD 94 (2016)
Mezrag, Segovia, Chang, Roberts, PLB 783 (2018)
Bednar, Cloet, Tandy, PRL 124 (2020)
Ding et al., PRD 101 (2020)
Serna et al., EPJ C 80 (2020)
Freese, Cloet, PRC 103 (2021)
Zhang et al., PLB 815 (2021)
Ydrefors, Frederico, PRD 104 (2021)
...

Parton distributions



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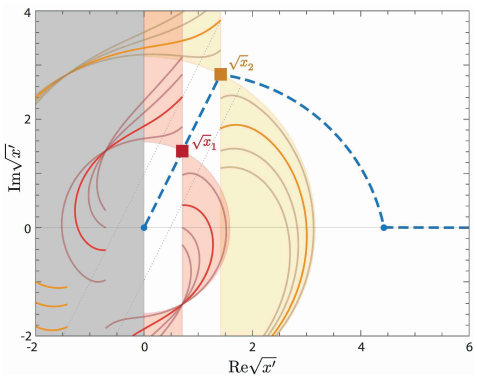
$$\Psi(z,P) = \langle 0|\mathsf{T}\,\Phi(z)\,\Phi(0)|P\rangle$$

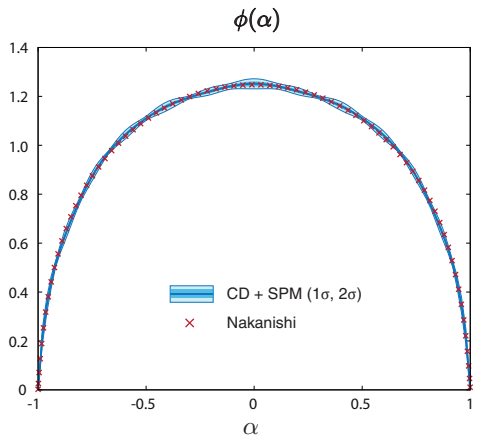
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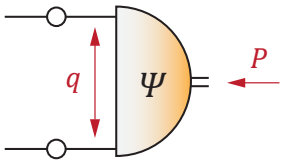
Novel method to compute
light-front wave functions
via contour deformations

Editors' Suggestion:
GE, Ferreira, Stadler, PRD 105 (2022)



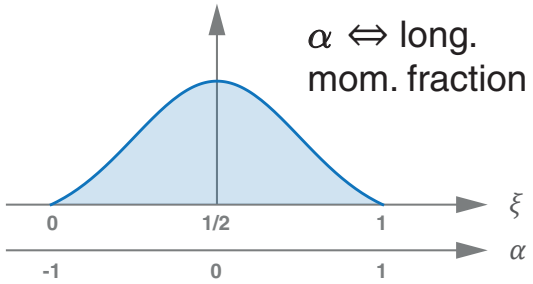


Light-front wave function

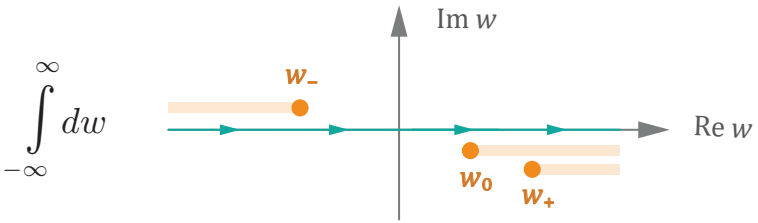


LFWF = BSWF integrated over q^-

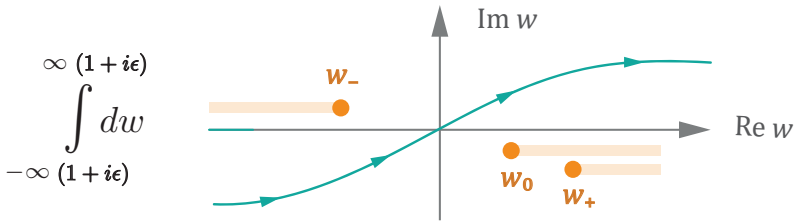
$$\psi(\alpha, \boldsymbol{k}_\perp) = \mathcal{N} P^+ \int_{-\infty}^{\infty} \frac{dq^-}{2\pi} \Psi(q, P)\big|_{q^+ = \frac{\alpha}{2} P^+,\, \boldsymbol{q}_\perp = \boldsymbol{k}_\perp}$$



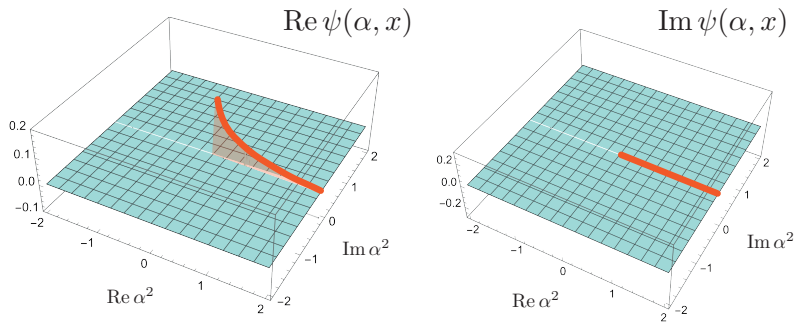
How is this related to an **analytic function**? Set $x = \frac{\boldsymbol{k}_\perp^2}{m^2}$, $w = \frac{M}{2m} q^-$, $t = -\frac{M^2}{4m^2}$

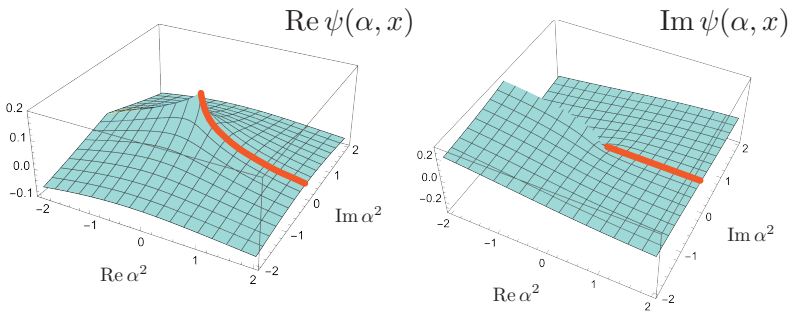


⇒ support only for $-1 < \alpha < 1$,
not an analytic function

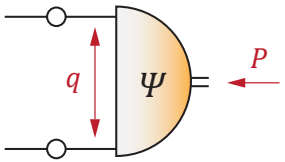


⇒ analytic function in α^2 for any x, t ∈ ℂ,
for $-1 < \alpha < 1$ result is the same



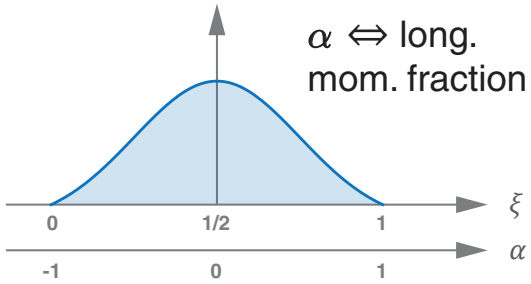


Light-front wave function

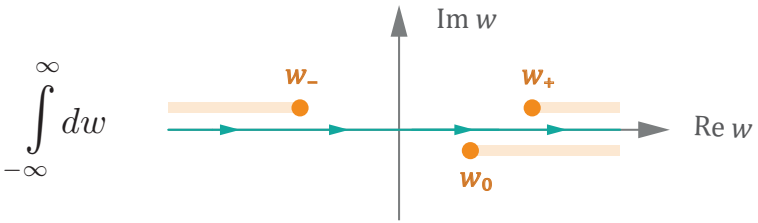


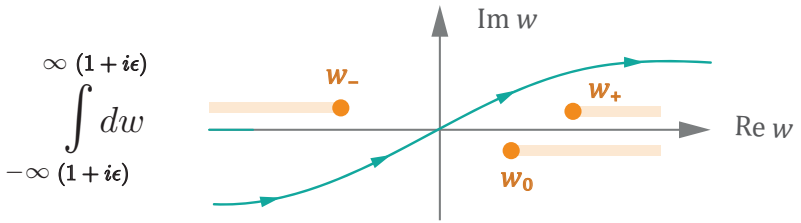
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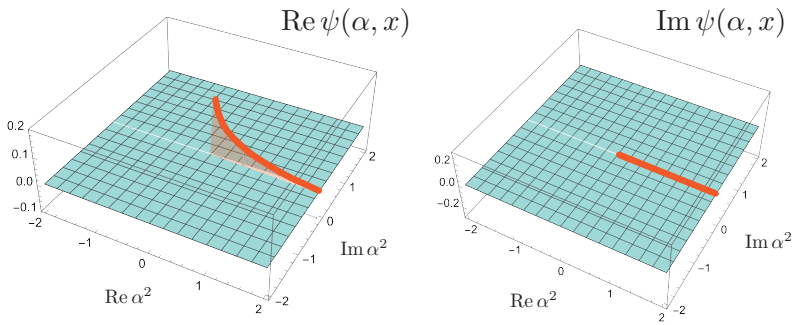
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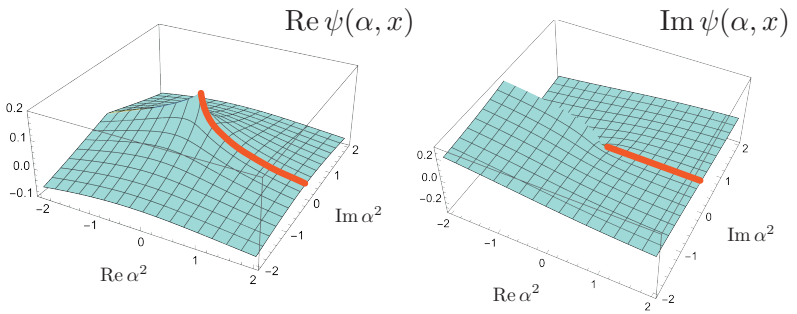




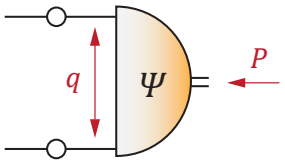
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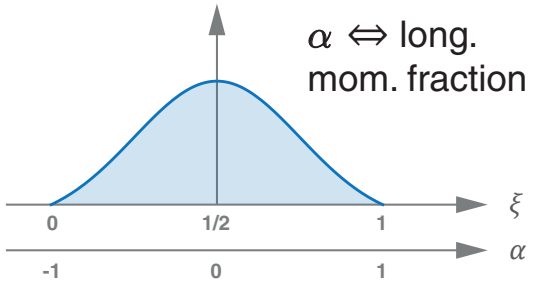


Light-front wave function

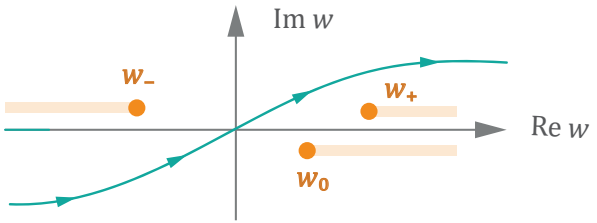


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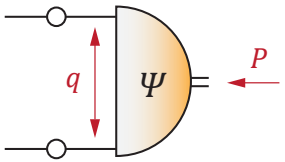


$$\psi(\alpha, x, t) = \frac{\mathcal{N} m^2}{i\pi} 2\sqrt{xt} \int_{-\infty}^{\infty} d\omega\, \Psi(x, \omega, t, \alpha)$$

$$\phi(\alpha) = \frac{m^2}{(4\pi)^2 f} \int_0^{\infty} dx\, \psi(\alpha, x, t)$$

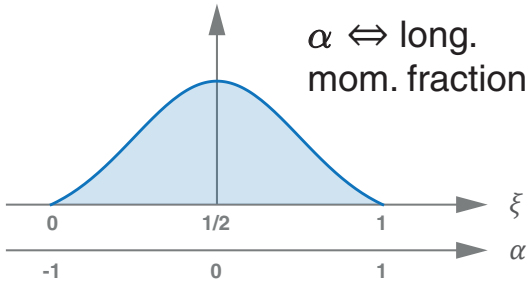
- Can be integrated **numerically** (check for monopole: same result)
- BSWF is **Lorentz-invariant**, can be calculated in any frame (also rest frame)
- But now the **branch cuts** in complex x plane will look different, need to stay inside $\mathcal{R}_\epsilon$

Light-front wave function



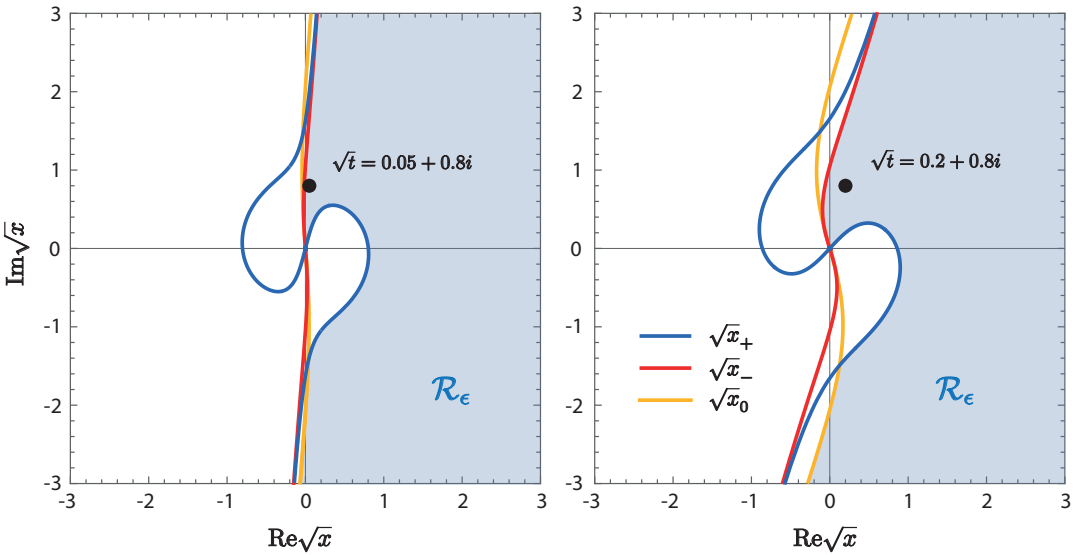
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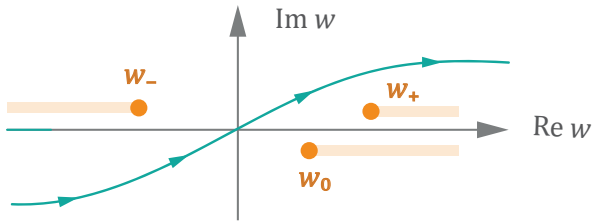
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After integration ⇒ **branch cuts** in complex $\sqrt{x}$ plane



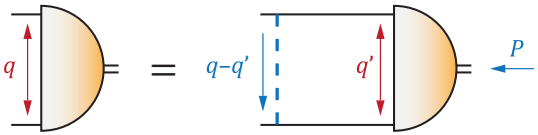


Propagator poles

Pole in BS amplitude

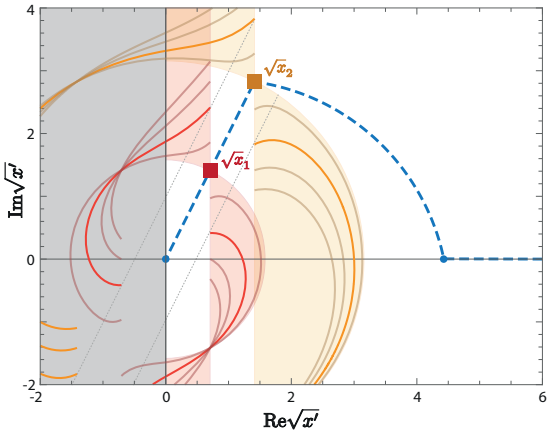
- Correct result for LFWF inside $\mathcal{R}_\epsilon$
- Physical region 0 < M < 2m is imaginary axis: $\sqrt{t} = \frac{iM}{2m}$
  Not possible ⇒ complex $\sqrt{t}$

Light-front wave function

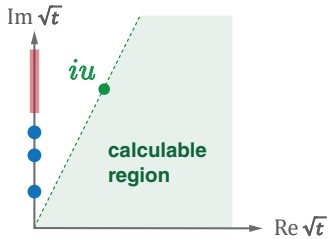


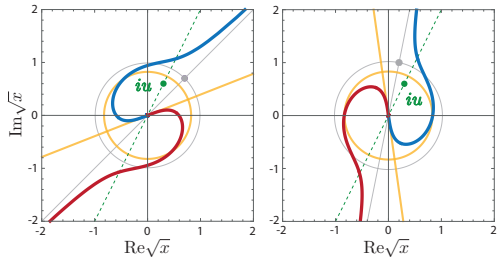
$$\Gamma(x,\omega,t,\alpha) = \int_0^\infty dx' \int_{-1}^{1} d\omega' \int_{-1}^{1} dy\, K(x,x',\Omega)\, G_0(x',\omega',t,\alpha)\, \Gamma(x',\omega',t,\alpha)$$

- BSE must be solved for $\sqrt{x} \in \mathcal{R}_\epsilon$
- BSE is **integral equation** ⇒ must be solved along path $\sqrt{x}$ that coincides with **integration path** $\sqrt{x}'$, must lie inside $\mathcal{R}_\epsilon$ ⇒ need contour deformations
- **Kernel** has pole ⇒ after integrating over y and ω', becomes branch cut in $\sqrt{x}'$, automatically avoided if $\mathrm{Re}\sqrt{x}'$ and $|\sqrt{x}'|$ increase along integration path

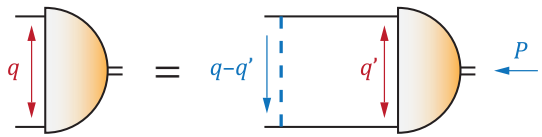


- **Propagators** have poles ⇒ branch cuts from before, automatically avoided if path within $\mathcal{R}_\epsilon$
- **BS amplitude** may dynamically generate singularities ⇒ avoided as long as arg $\sqrt{t}$ < arg(iu)



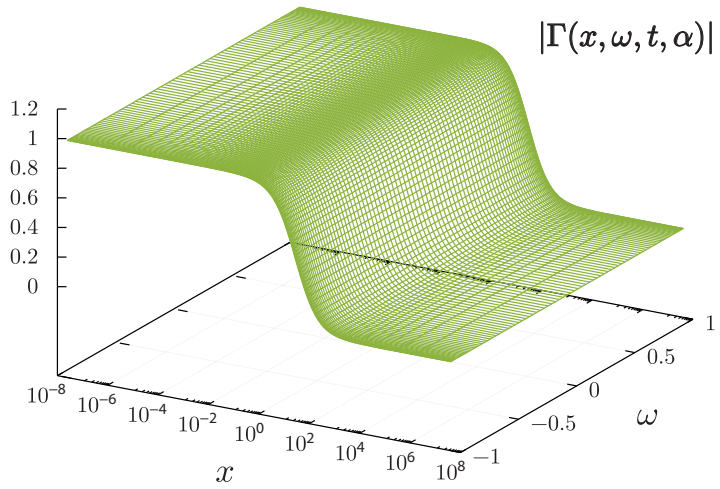


Light-front wave function



$$\Gamma(x,\omega,t,\alpha) = \int_0^\infty dx' \int_{-1}^{1} d\omega' \int_{-1}^{1} dy\, K(x,x',\Omega)\, G_0(x',\omega',t,\alpha)\, \Gamma(x',\omega',t,\alpha)$$

$$\Omega = \omega\omega' + y\sqrt{1-\omega^2}\sqrt{1-\omega'^2}$$



$\alpha = 0.6$,
$\sqrt{t} = 0.2 + 0.2i$

- BS amplitude falls off like 1/x
- only weak dependence on ω and α
  $\Rightarrow \alpha$ dependence in LFWF comes from propagators:

$$\psi(\alpha,x,t) \propto \int_{-\infty}^{\infty} d\omega\, G_0(x,\omega,t,\alpha)\, \Gamma(x,\omega,t,\alpha)$$

- LFWF needs Γ for $\omega \in (-\infty,\infty)$, but
  BSE solution only known for $\omega \in [-1,1]$

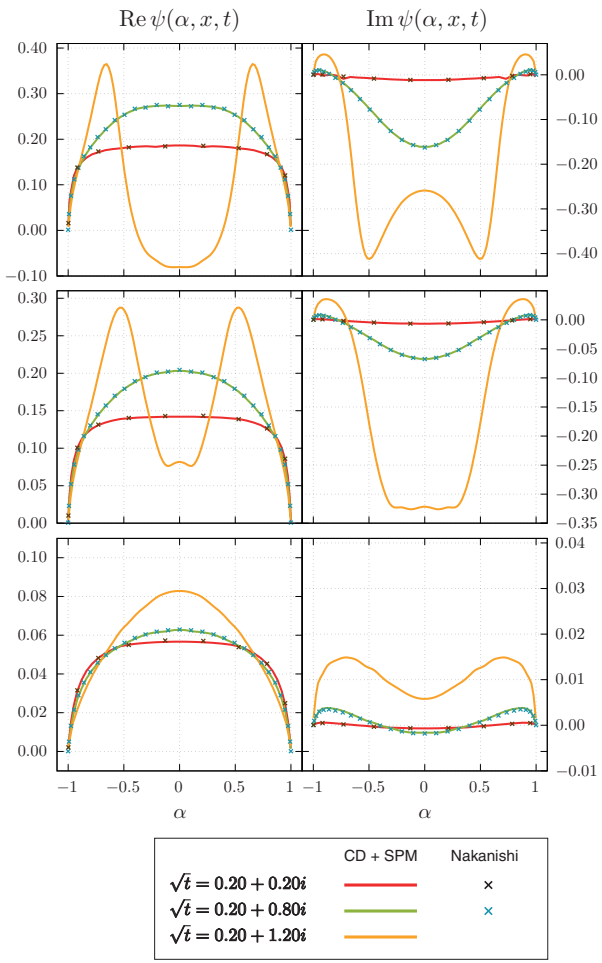
  Use **Schlessinger point method** (SPM)
  for analytic continuation

  Schlessinger, Phys. Rev. 167 (1968), Tripolt, Haritan, Wambach, Moiseyev, PLB 774 (2017), Binosi, Tripolt, PLB 801 (2020), ...

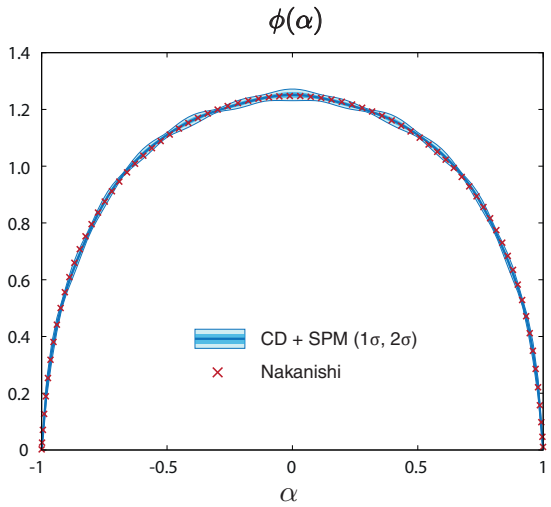
$$f(\omega) = \cfrac{c_1}{1 + \cfrac{c_2\,(\omega-\omega_1)}{1 + \cfrac{c_3\,(\omega-\omega_2)}{1 + \cfrac{c_4\,(\omega-\omega_3)}{\cdots}}}}$$

Light-front wave function

Light-front wave function
(IR, mid-momentum, UV)



Parton distribution amplitude



On the physical axis:
$\sqrt{t} = 0.5\mathrm{i}$

use Schlessinger method in t

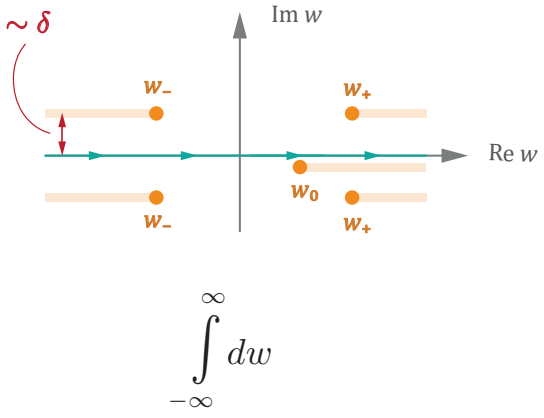
GE, Ferreira, Stadler, PRD 105 (2022)

- Results agree with Nakanishi method — Frederico, Salmè, Viviani, PRD 85 (2012)
- LFWF vanishes at endpoints $\alpha = \pm 1$
- No expansion in moments involved, plain numerical result
- Also works above threshold (unphysical, no poles on 1st sheet, no resonances either)

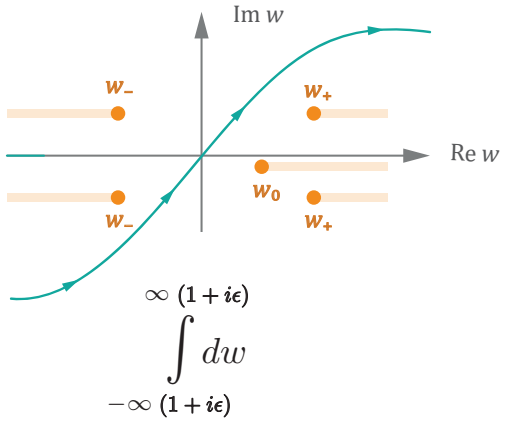
Complex conjugate singularities

Straightforward to implement in scalar model, contour deformations work in same way

$$D(q^2) = \frac{1}{2}\left(\frac{1}{q^2+m^2\,(1+i\delta)} + \frac{1}{q^2+m^2\,(1-i\delta)}\right) = \frac{q^2+m^2}{(q^2+m^2)^2+m^4\,\delta^2}$$



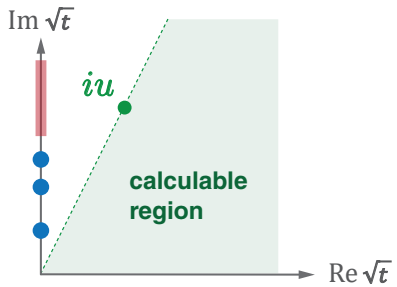
does not give correct limit for $\delta \to 0$



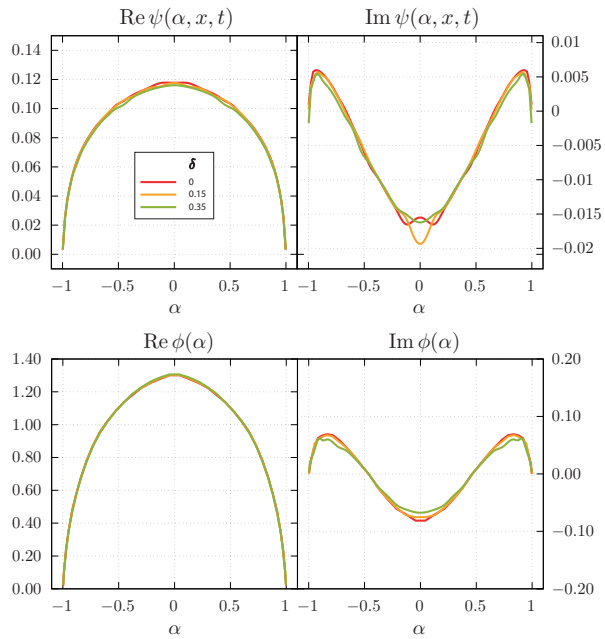
correct limit for $\delta \to 0$, proper analytic continuation

- One does not even need to know pole positions!

  ⇒ works for general singularities in n-point functions!



LFWF and PDA not sensitive to δ :

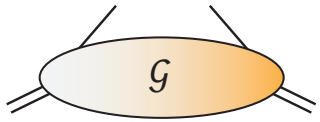


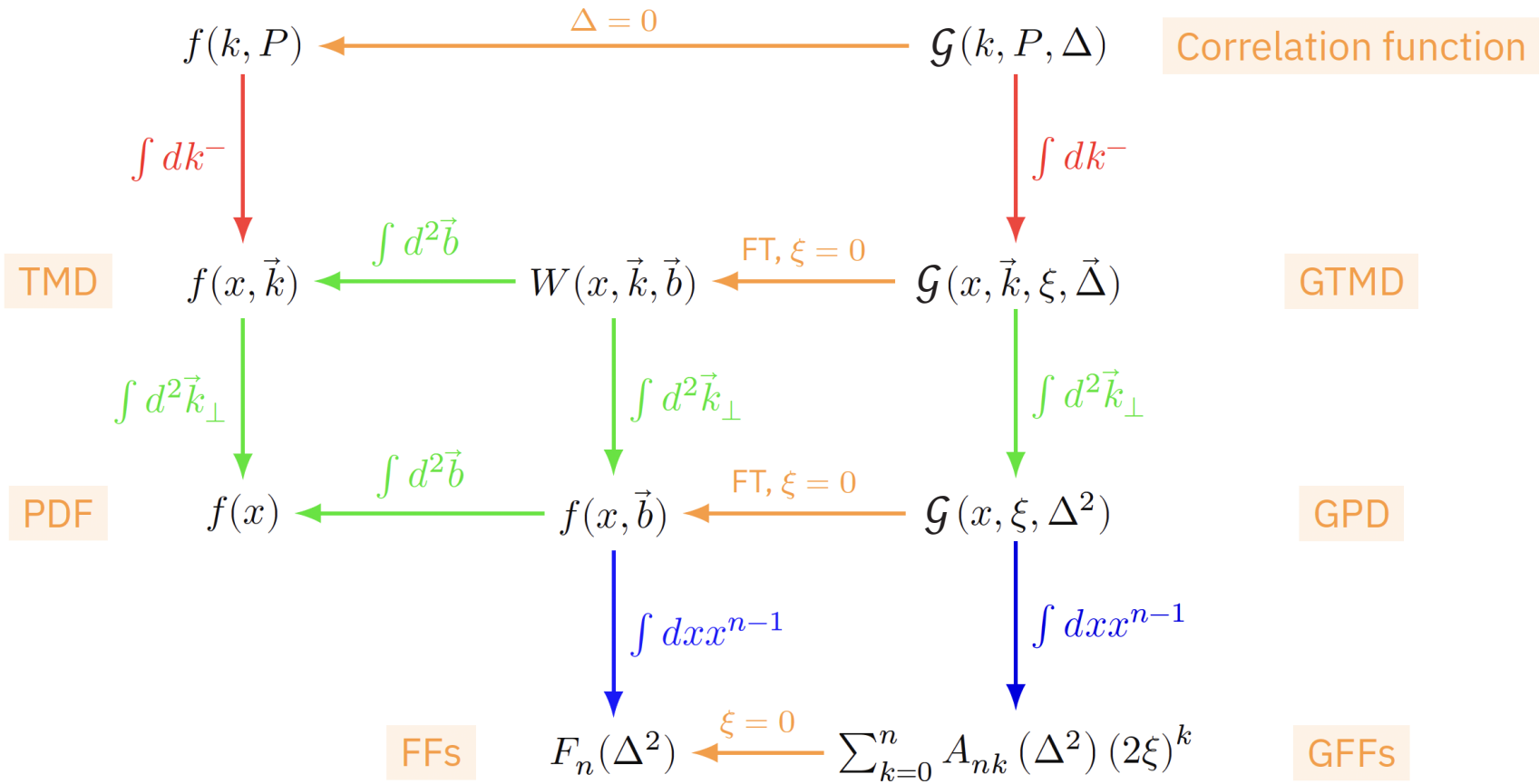
GE, Ferreira, Stadler, PRD 105 (2022)

PDFs, TMDs, GPDs?

All generated from hadron-to-hadron correlator

Diehl 2003, Lorcé, Pasquini, Vanderhaeghen 2011,
Meißner, Goeke, Metz, Schlegel 2008, Meißner, Metz, Schlegel 2009





Eduardo Ferreira, ACHT 2025

Thank you!

Backup slides

Gluon mass generation

Gluon propagator:

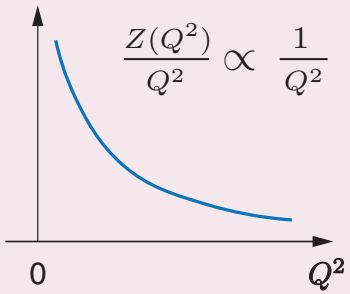


transverse dressing

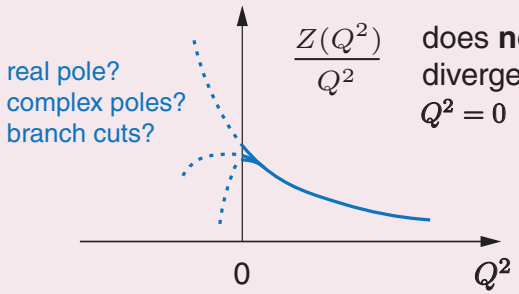
longitudinal dressing = 1

$$D^{\mu\nu}(Q) = \frac{Z(Q^2)}{Q^2}\left(\delta^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2}\right) + \xi\,\frac{L(Q^2)}{Q^2}\,\frac{Q^\mu Q^\nu}{Q^2}$$

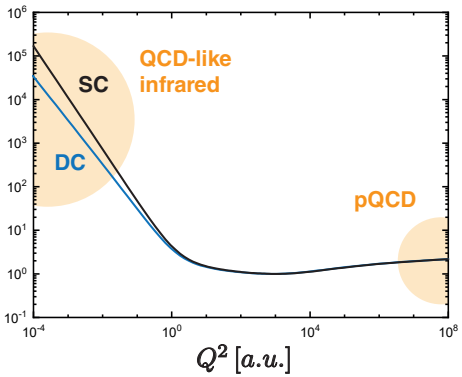
- **Perturbation theory:** Massless gluon pole



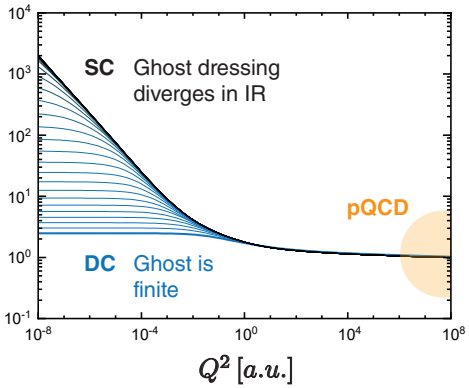
- **Nonperturbative calculations:** Massless pole disappears!



Gluon $1/Z(Q^2)$



Ghost $G(Q^2)$



Family of **"decoupling"** solutions, also seen in lattice QCD

Cucchieri, Maas, Mendes, PRD 77 (2008)
Aguilar, Binosi, Papavassiliou, PRD 78 (2008)
Boucaud et al., JHEP 06 (2008)
Bogolubsky et al., PLB 676 (2009)
Fischer, Maas, Pawlowski, Ann. Phys. 324 (2009)
Duarte, Oliveira, Silva, PRD 94 (2016)
Aguilar et al., EPJ C 80 (2020)

Endpoint is **"scaling"** solution, confinement manifest

Lerche, Smekal, PRD 65 (2002)
Fischer, Alkofer, PLB 536 (2002)
Alkofer, Fischer, Llanes-Estrada, MPLA 23 (2008)

All solutions show **gluon mass gap**

$$\lim_{r\to\infty} \int \frac{d^3Q}{(2\pi)^3}\,\frac{Z(Q^2)}{Q^2}\,e^{i\,\boldsymbol{x}\cdot\boldsymbol{Q}} \propto e^{-m_{\rm gap}\,r}$$

Schwinger mechanism:
Gluon correlation functions develop massless longitudinal poles

Aguilar, Ibanez, Mathieu, Papavassiliou, PRD 85 (2012)
Aguilar, Binosi, Papavassiliou, PRD 95 (2017)
GE, Pawlowski, Silva, PRD 104 (2021)
Aguilar, Ferreira, Papavassiliou, PRD 105 (2022)

Gluon mass generation

$$(D^{-1})^{\mu\nu}(Q) = (D_0^{-1})^{\mu\nu}(Q) + \Pi^{\mu\nu}(Q)$$



$$Z(Q^2)^{-1} = Z_A + \Delta_T + \frac{\Delta_0}{Q^2}$$

$$L(Q^2)^{-1} = 1 + \xi \left[\Delta_L + \frac{\Delta_0}{Q^2}\right] \overset{!}{=} 0$$

Expand self-energy in overcomplete basis:

$$\Delta_T(Q^2)\,(Q^2\delta^{\mu\nu} - Q^\mu Q^\nu) + \Delta_0(Q^2)\,\delta^{\mu\nu} + \Delta_L(Q^2)\,Q^\mu Q^\nu$$

contains **quadratic divergences,** need to be subtracted

must contain **longitudinal massless poles**

$$\Delta_L = -\frac{\Delta_0}{Q^2}$$

Mass generation must come from here!

⇒ Gauge consistency needs **longitudinal massless poles** in vertices



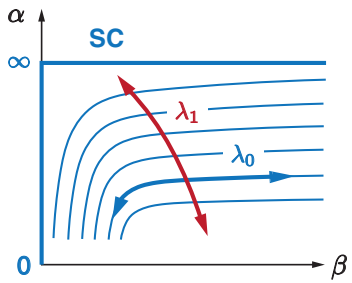
ghost-gluon vertex?

three-gluon vertex?

four-gluon vertex?

- YM solutions depend on 2 parameters: coupling α, mass parameter $\beta \Leftrightarrow \Delta_0$

GE, Pawlowski, Silva, PRD 104 (2021)



$\lambda_0(\alpha, \beta)$ only rescales solutions,
$\lambda_1(\alpha, \beta)$ distinguishes SC & DC

- SC solution: ghost-gluon vertex has long. massless pole

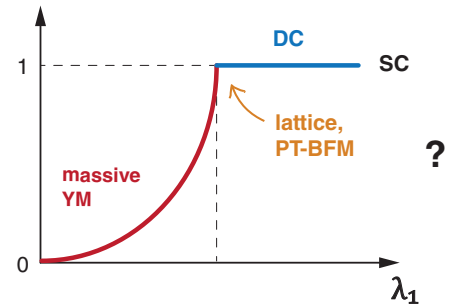
GE, Pawlowski, Silva, PRD 104 (2021)

- DC solution: three-gluon vertex has long. massless pole

Aguilar, Ibanez, Mathieu, Papavassiliou, PRD 85 (2012),
Ferreira, Papavassiliou, Particles 6 (2023)

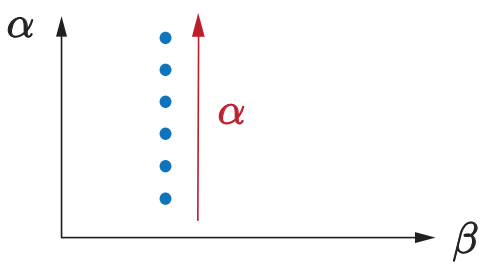
- SC & DC physically equivalent? Test in hadron observables!

BSE eigenvalue
(=1 means: long. massless pole)



Family of solutions

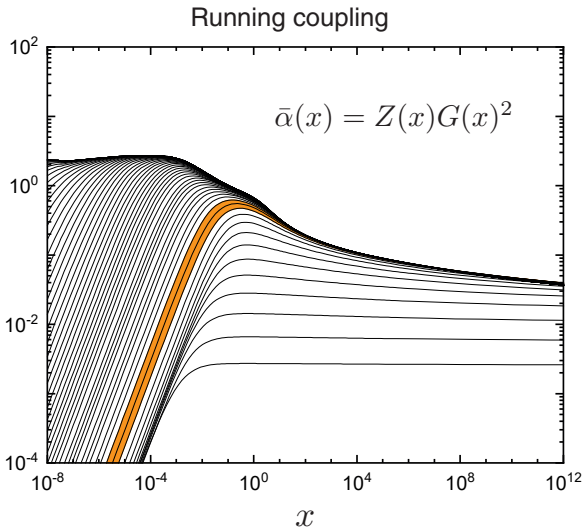
Fixed β, vary $\alpha \in (0, \infty)$:

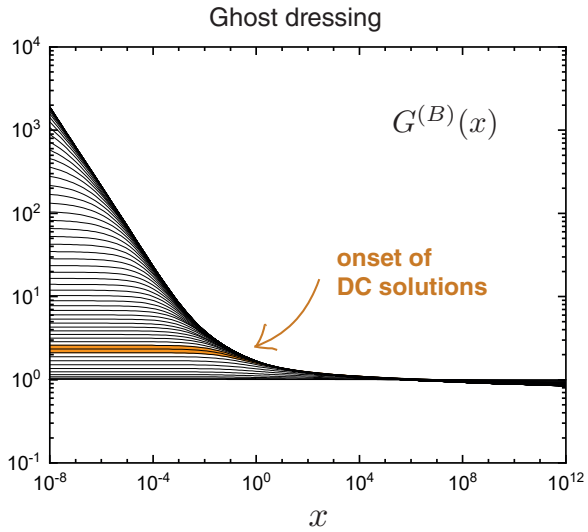


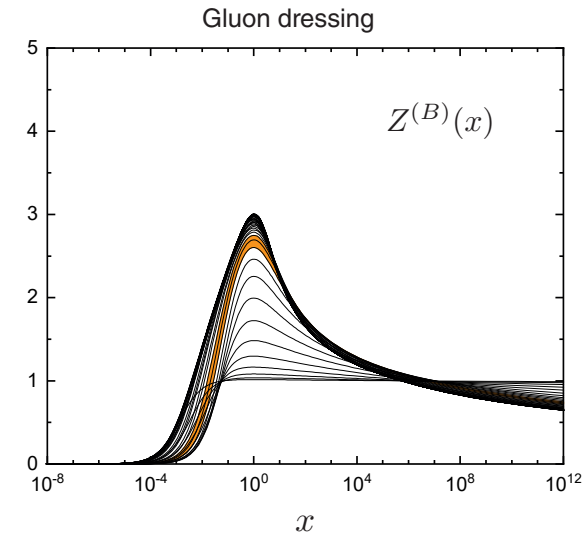
- **Running coupling** $\bar{\alpha}(x)$ is RG-invariant, sets scale (so far x is arbitrary)
- SC = envelope of DC solutions for $\alpha \to \infty$

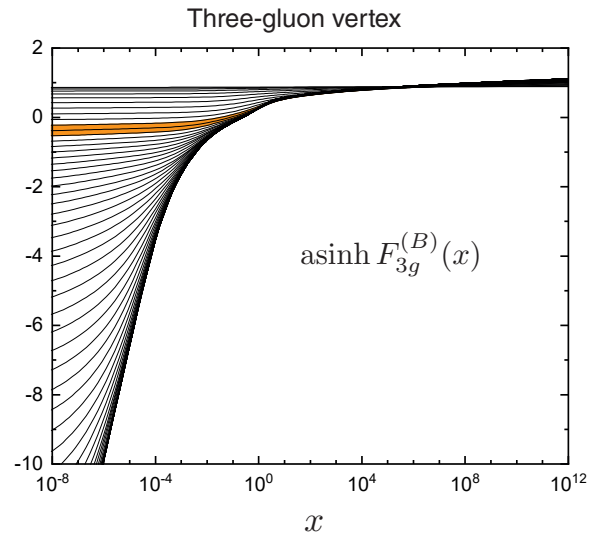
$$Z \sim (Q^2)^{2\kappa}$$
$$G \sim (Q^2)^{-\kappa}$$
$$F_{3g} \sim (Q^2)^{-3\kappa}$$
$$\bar{\alpha} \sim \text{ const.}$$

- Onset of DC solutions ↔ **lattice** solutions







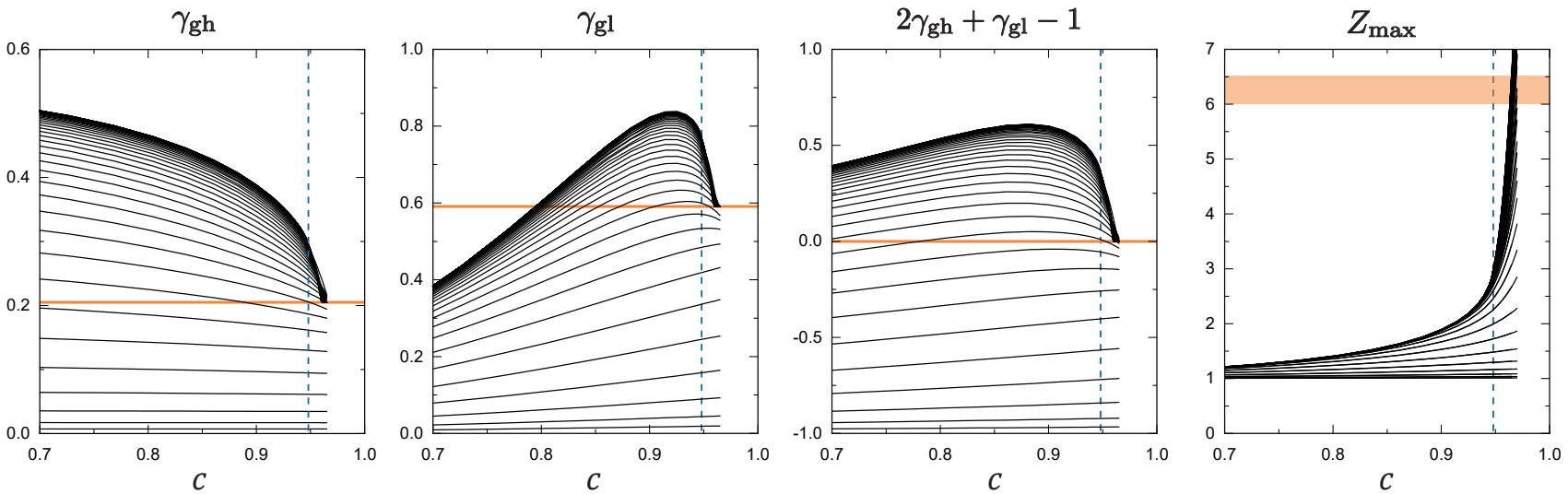


Truncation error

- Set $Z_{3g} \to c\, Z_{3g}$ ... quantifies deviation from STI (without truncation: $c = 1$), same effect from "over-renormalizing" 3-gluon vertex
- YM system only converges up to $c_{\max} < 1$
- Anomalous dimensions reproduced for
  - 1 c ~ 0.4
  - 2 c ~ 0.9
  - 3 c ~ 0.96

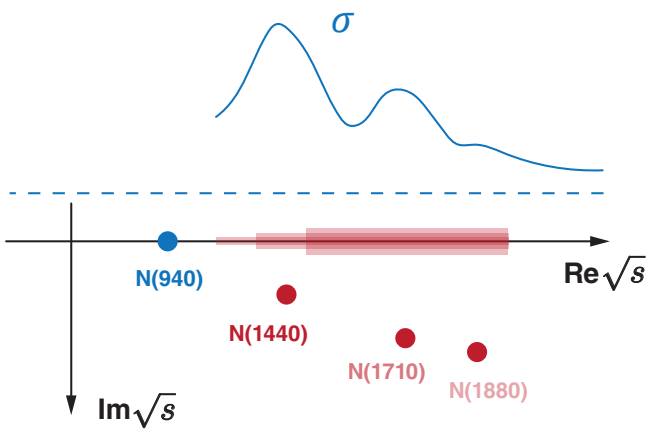
  $\Rightarrow$ identifies "physical point" for each truncation

  GE, Pawlowski, Silva, PRD 104 (2021)



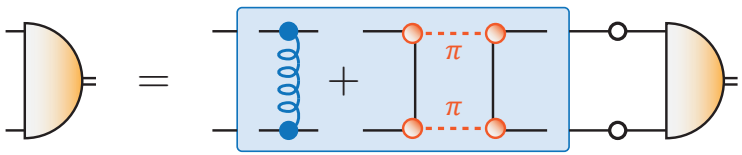
Resonances

- Most hadrons are **resonances** and decay ⇔ poles in complex momentum plane



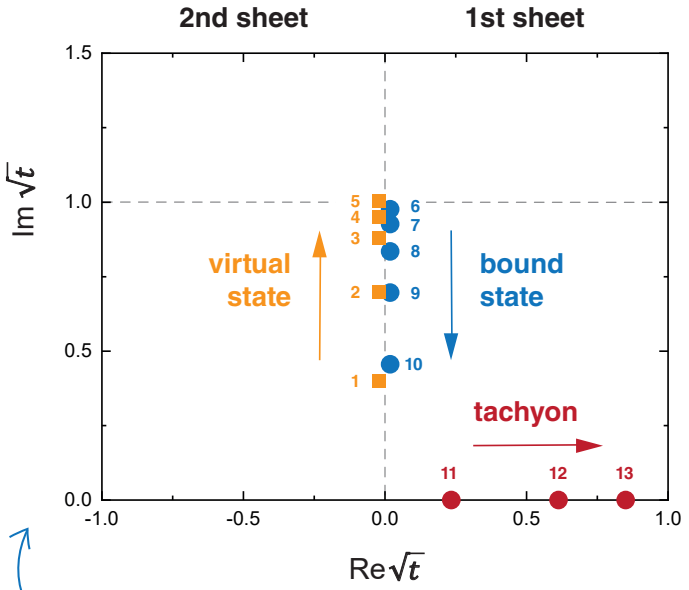
- BSE kernel must include decay channels: ρ meson becomes resonance

  Williams, PLB 798 (2019), Miramontes, Sanchis-Alepuz, EPJA 55 (2019), Santowsky, GE, Fischer, Wallbott, PRD 102 (2020), Miramontes, Sanchis-Alepuz, Alkofer, PRD 103 (2021)

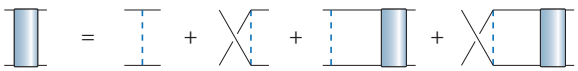


- **Contour deformations** as tool to go beyond thresholds

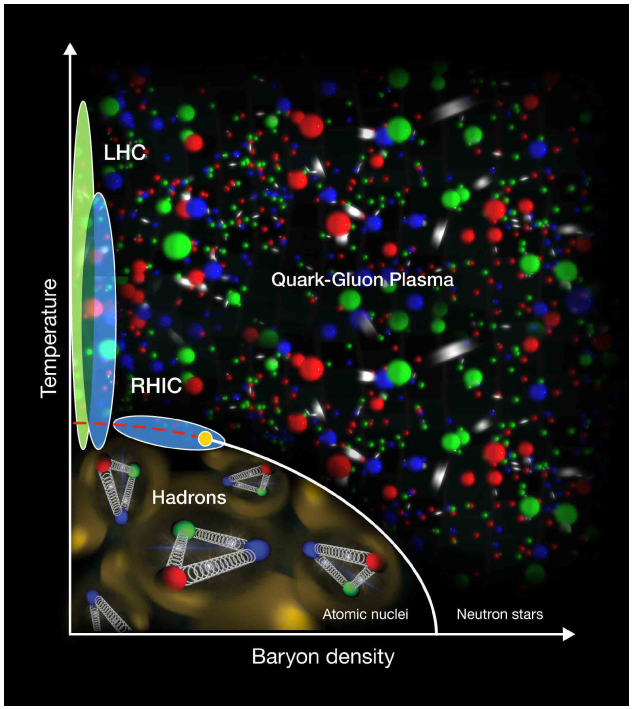
  GE, Duarte, Peña, Stadler, PRD 100 (2019)



Scattering equation for 4-point function

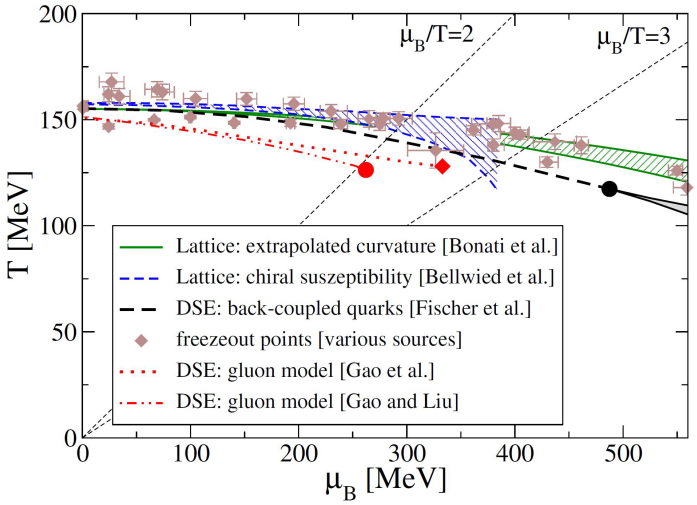


QCD phase diagram



Brookhaven National Laboratory

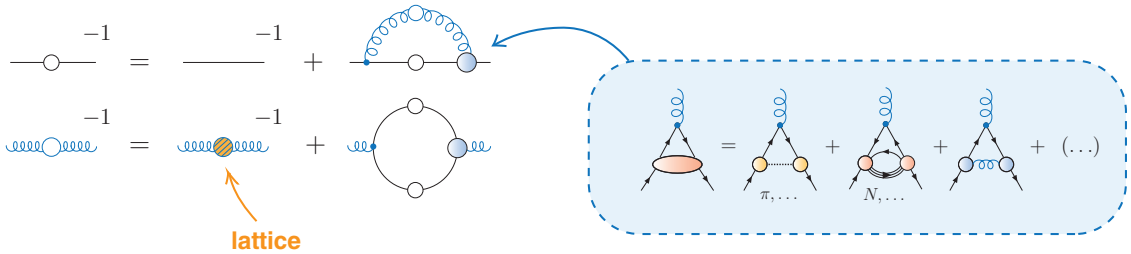
Search for **critical endpoint (CEP)** from DSEs & lattice:



Fischer, Prog. Part. Nucl. Phys. 105 (2019)

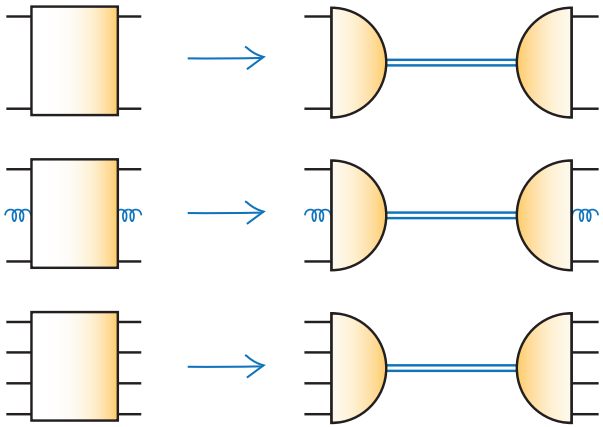
Location of CEP sensitive to **baryons**?

GE, Fischer, Welzbacher, PRD 93 (2016)

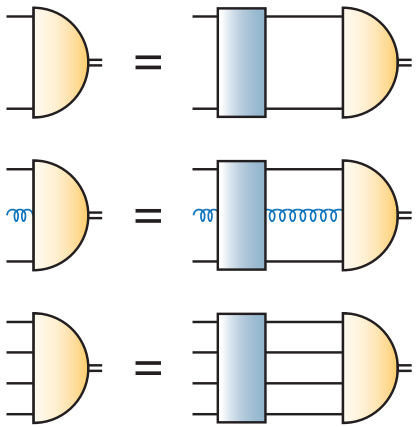


Some thoughts

- Same spectral representation for **all** correlation functions that produce $q\bar{q}$:



- **Without** truncations, BSEs for qq, $qq\bar{q}\bar{q}$, $q\bar{q}g$, ... should produce **same spectrum**



→ harder to include $qq\bar{q}\bar{q}$, $q\bar{q}g$

→ harder to include $q\bar{q}$ components

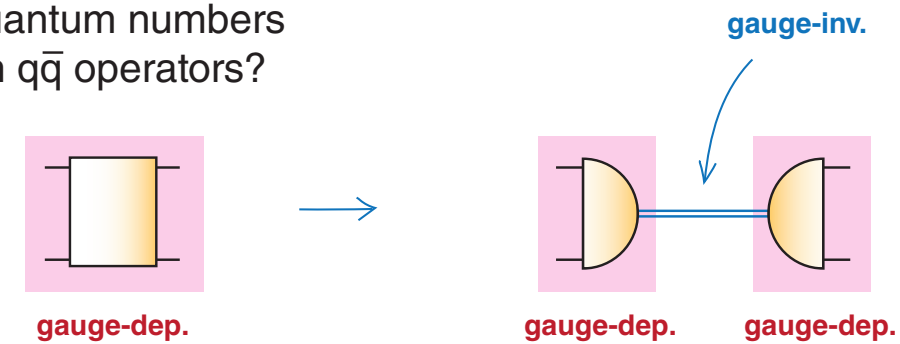
With truncations, $q\bar{q}$ ($qq\bar{q}\bar{q}$, $q\bar{q}g$, ...) BSE should give more reliable spectrum for $q\bar{q}$ ($qq\bar{q}\bar{q}$, $q\bar{q}g$, ...) dominated states

Exotic quantum numbers not excluded in principle (need gluon-rich kernel in $q\bar{q}$ BSE)

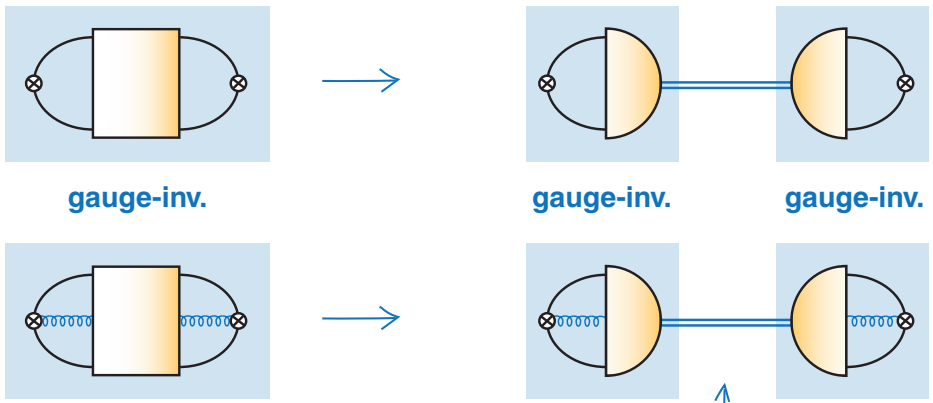
Some thoughts

- Why don't we see exotic quantum numbers in **lattice calculations** with $q\bar{q}$ operators?

$\langle q_\alpha \, \bar{q}_\beta \, q_\gamma \, \bar{q}_\delta \rangle \; =$



$\langle (\bar{q} \, \Gamma q)(\bar{q} \, \Gamma q) \rangle \; =$



= 0 for exotic quantum nrs (?)

≠ 0 for exotic quantum nrs

Could test with **gauge-fixed** lattice calculations: same spectrum?